Planar System $x' = Ax$

- Equilibrium points for the system
- Set of equilibrium points equals null($A$).
- $A$ nonsingular $\Rightarrow$ only equilibrium point is 0.
- Can we list the types of all possible equilibrium points for planar linear systems?
  - We will do the six most important cases.
  - The other cases are Project #3.
  - Look at solution curves in the phase plane.

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.
  \[ \lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2} \]
- Eigenvectors $v_1$ and $v_2$. General solution
  \[ x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2 \]
- $\lambda_1 < 0 < \lambda_2$ Saddle point.
- $\lambda_1 < \lambda_2 < 0$ Nodal sink.
- $0 < \lambda_1 < \lambda_2$ Nodal source.
Complex Eigenvalues
- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D < 0$
  $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$.
- Eigenvector $w = v_1 + iv_2$ associated to $\lambda$.
- Complex solutions
  \[ z(t) = e^{\lambda t}w = e^{(\alpha+i\beta)t}[v_1 + iv_2] \]
  \[ \bar{z}(t) = e^{\bar{\lambda} t} \bar{w} = e^{(\alpha-i\beta)t}[v_1 - iv_2] \]

Real solutions
- $x_1(t) = \text{Re}(z(t)) = e^{\alpha t}[\cos \beta t \cdot v_1 - \sin \beta t \cdot v_2]$
- $x_2(t) = \text{Im}(z(t)) = e^{\alpha t}[\sin \beta t \cdot v_1 + \cos \beta t \cdot v_2]$
- General solution
  \[ x(t) = C_1 e^{\alpha t}[\cos \beta t \cdot v_1 - \sin \beta t \cdot v_2] \]
  \[ + C_2 e^{\alpha t}[\sin \beta t \cdot v_1 + \cos \beta t \cdot v_2] \]

Center
- $\alpha = \text{Re}(\lambda) = 0$
- General real solution
  \[ x(t) = C_1[\cos \beta t \cdot v_1 - \sin \beta t \cdot v_2] \]
  \[ + C_2[\sin \beta t \cdot v_1 + \cos \beta t \cdot v_2] \]
- Every solution is periodic with period $T = 2\pi/\beta$.
- All solution curves are ellipses.
- Direction of flow can be found by looking at the vector field at one point. $(1, 0)^T$ is a good choice.
**Spiral Sink**

- $\alpha = \text{Re}(\lambda) < 0$
- General real solution
  \[ x(t) = C_1 e^{\alpha t} \left[ \cos \beta t \cdot v_1 - \sin \beta t \cdot v_2 \right] + C_2 e^{\alpha t} \left[ \sin \beta t \cdot v_1 + \cos \beta t \cdot v_2 \right] \]
- All solutions spiral into 0 as $t \to \infty$.

**Spiral Source**

- $\alpha = \text{Re}(\lambda) > 0$
- General real solution
  \[ x(t) = C_1 e^{\alpha t} \left[ \cos \beta t \cdot v_1 - \sin \beta t \cdot v_2 \right] + C_2 e^{\alpha t} \left[ \sin \beta t \cdot v_1 + \cos \beta t \cdot v_2 \right] \]
- All solutions spiral into 0 as $t \to -\infty$.

**Planar Systems**

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

- Char. polynomial $p(\lambda) = \lambda^2 - T\lambda + D$.
- Eigenvalues
  \[ \lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2} \]
• $\lambda_1$ and $\lambda_2$ are the roots of $p(\lambda)$, so

\[
p(\lambda) = \lambda^2 - T\lambda + D = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2
\]

• $T = \lambda_1 + \lambda_2$ and $D = \lambda_1\lambda_2$.
• Duality between $(\lambda_1, \lambda_2)$ and $(T, D)$.
• Represent systems by location of $(T, D)$ in the $TD$-plane.

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Trace-Determinant Plane

• $T^2 - 4D > 0$
  • $\Rightarrow$ distinct real eigenvalues $\lambda_1$ and $\lambda_2$
  • $D = \lambda_1\lambda_2 < 0 \Rightarrow$ Saddle point.
  • $D = \lambda_1\lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
    • $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
    • $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

• $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues
  \[
  \lambda = \alpha + i\beta \quad \text{and} \quad \bar{\lambda} = \alpha - i\beta.
  \]
  • $T = \lambda + \bar{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.
  • $T = \lambda + \bar{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.
  • $T = \lambda + \bar{\lambda} = 2\alpha = 0 \Rightarrow$ Center.
Types of Equilibrium Points

- **Generic types**
  - Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.

- **Nongeneric types**
  - Center and many others. Occupy pieces of the boundaries between the generic types.
  - Project #3 asks you to find the rest of the types.

Higher Dimensional Systems

\[ x' = Ax \]

- \( A \) is a real \( n \times n \) matrix.
- If \( \lambda \) is an eigenvalue and \( v \neq 0 \) is an associated eigenvector, then \( x(t) = e^{\lambda t}v \) is a solution.
- Much like the planar case, but now we need \( n \) linearly independent solutions.
- We no longer have the easy way to compute the characteristic polynomial \( p(\lambda) = \det(A - \lambda I) \).

**Proposition:** Suppose that \( \lambda_1, \ldots, \lambda_k \) are distinct eigenvalues of \( A \), and that \( v_1, \ldots, v_k \) are associated nonzero eigenvectors. Then \( v_1, \ldots, v_k \) are linearly independent.

**Theorem:** Suppose the \( n \times n \) real matrix \( A \) has \( n \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \), and that \( v_1, \ldots, v_n \) are associated nonzero eigenvectors. Then the exponential solutions \( x_i(t) = e^{\lambda_i t}v_i, 1 \leq i \leq n \) form a fundamental set of solutions for the system \( x' = Ax \).
Example

\[ A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix} \]

- Use MATLAB.

Complex Eigenvalues

A real \( n \times n \) matrix with a complex eigenvalue \( \lambda \) and associate eigenvector \( w \).

- \( \Rightarrow \lambda \) is an eigenvalue and \( w \) is an associated nonzero eigenvector.
- Complex valued solutions: \( z(t) = e^{\lambda t}w \)
  \[ x(t) = e^{\lambda t}w, \]
- Real solutions: \( x(t) = \text{Re}(z(t)) \)
  \[ y(t) = \text{Im}(z(t)). \]

Example

\[ A = \begin{pmatrix} 21 & 10 & 4 \\ -70 & -31 & -10 \\ 30 & 10 & -1 \end{pmatrix} \]

- The theorem applies if some of the eigenvalues are complex and we replace complex conjugate pairs of solutions by their real and imaginary parts.
Repeated Eigenvalues – Example 1

\[ A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix} \]

- \( p(\lambda) = (\lambda + 3)(\lambda + 1)^2 \)
- \( \lambda_1 = -3 \)
  - Eigenspace has dimension 1 \( \Rightarrow \) one exponential solution
  \[ x_1(t) = e^{-3t}(-1/3, 2/3, 1)^T \]

- \( \lambda_2 = -1 \)
  - Eigenspace has dimension 2 \( \Rightarrow \) two linearly independent exponential solutions
  - Eigenspace has basis \( v_2 = (-5/2, 1, 0)^T \) and \( v_3 = (3/2, 0, 1)^T \).
  - Linearly independent solutions
  \[ x_2(t) = e^{-t} \begin{pmatrix} -5/2 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad x_3(t) = e^{-t} \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix} \]
  - \( x_1, x_2, \) and \( x_3 \) are a fundamental set of solutions.

Repeated Eigenvalues – Example 2

\[ A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix} \]

- \( p(\lambda) = (\lambda + 3)(\lambda + 1)^2 \)
- \( \lambda_1 = -3 \)
  - Eigenspace has dimension 1 \( \Rightarrow \) one exponential solution
  \[ x_1(t) = e^{-3t}(-1/2, 3/2, 1)^T \]
• $\lambda_2 = -1$
• Eigenspace has dimension 1 $\Rightarrow$ only one exponential solution

$$x_2(t) = e^{-t} \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}$$

• Need a third solution.
• Need a new idea.

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### Multiplicities

A an $n \times n$ matrix

• Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
• The characteristic polynomial is

$$p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}.$$  

• The algebraic multiplicity of $\lambda_j$ is $q_j$.
• The geometric multiplicity of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$.

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• We always have:
  • $q_1 + q_2 + \cdots + q_k = n$.
  • $1 \leq d_j \leq q_j$.
  • There are $d_j$ linearly independent exponential solutions corresponding to $\lambda_j$.
  • If $d_j = q_j$ for all $j$ we have $n$ linearly independent solutions.
  • If $d_j < q_j$ we have trouble.