Math 211
Lecture #30
The Exponential of a Matrix

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Repeated Eigenvalues – Example 1

\[ A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix} \]

- \( p(\lambda) = (\lambda + 3)(\lambda + 1)^2 \)
- \( \lambda_1 = -3 \) : Eigenspace has dimension 1, with basis \( v_1 \), so there is one exponential solution, \( x_1(t) = e^{\lambda_1 t} v_1 \).
- \( \lambda_2 = -1 \) : Eigenspace has dimension 2 with basis \( v_2 \) and \( v_3 \), so there are two linearly independent exponential solutions \( x_2(t) = e^{\lambda_2 t} v_2 \) and \( x_3(t) = e^{\lambda_2 t} v_3 \).
- \( x_1, x_2, \) and \( x_3 \) are a fundamental set of solutions.

Repeated Eigenvalues – Example 2

\[ A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix} \]

- \( p(\lambda) = (\lambda + 3)(\lambda + 1)^2 \)
- \( \lambda_1 = -3 \)
- Eigenspace has dimension 1 \( \Rightarrow \) one exponential solution
  \[ x_1(t) = e^{-3t}(-1/2, 3/2, 1)^T \]
• \( \lambda_2 = -1 \)
  - Eigenspace has dimension 1 \( \Rightarrow \) only one exponential solution
  \[ x_2(t) = e^{-t} \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix} \]
  - Need a third solution.
  - Need a new idea.

### Multiplicities

A an \( n \times n \) matrix
- Distinct eigenvalues \( \lambda_1, \ldots, \lambda_k \).
- The characteristic polynomial is
  \[ p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}. \]
  - The algebraic multiplicity of \( \lambda_j \) is \( q_j \).
  - The geometric multiplicity of \( \lambda_j \) is \( d_j \), the dimension of the eigenspace of \( \lambda_j \).

- We always have:
  - \( q_1 + q_2 + \cdots + q_k = n \).
  - \( 1 \leq d_j \leq q_j \).
  - There are \( d_j \) linearly independent exponential solutions corresponding to \( \lambda_j \).
  - If \( d_j = q_j \) for all \( j \) we have \( n \) linearly independent solutions.
  - If \( d_j < q_j \) we have trouble.
New Approach
• $D = 1: x' = ax$
• Solution $x(t) = Ce^{at}$.
• $D > 1: x' = Ax$
  • Tried $x(t) = e^{At}v$.
  • Worked well except when eigenvalues have multiplicity greater than 1.
  • Why not $x(t) = e^{At}v$?
  • But what is $e^{At}$?

Exponential of a Matrix
Definition: The exponential of the $n \times n$ matrix $A$ is the $n \times n$ matrix
$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots$$
$$= \sum_{0}^{\infty} \frac{1}{n!}A^n.$$

• Example:
  • $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \Rightarrow e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}$.
  • $e^{At} = e^A \cdot e^{tI}$.  

Properties
• $A$ commutes with $e^A$, 
  $$Ae^A = e^A A.$$
• If $A$ and $B$ commute (i.e., $AB = BA$), then 
  $$e^{A+B} = e^A \cdot e^B.$$
• The inverse of $e^A$ is $e^{-A}$.
• $\frac{d}{dt}e^{tA} = Ae^{tA}$. 
Important Fact
Theorem: The solution to the initial value problem
\[ x' = Ax \quad \text{with} \quad x(0) = v \]
is \( x(t) = e^{tA}v \).
• However computing \( e^{tA} \) is not easy.

Key to Computing \( e^{tA} \) or \( e^{tA}v \)

Suppose that \( A \) an \( n \times n \) matrix, and \( \lambda \) a number (an eigenvalue).
• \( A = \lambda I + (A - \lambda I) \); (\( \lambda I \) & \( A - \lambda I \) commute.)
\[
e^{tA} = e^{t(\lambda I + (A - \lambda I))} \\
= e^{t\lambda I} \cdot e^{t(A - \lambda I)} \\
= e^{t\lambda} \cdot e^{t(\lambda - A)} \\
= e^{t\lambda} \cdot \left[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots\right]
\]

\( e^{tA}v \), \( v \) an Eigenvector

Let \( \lambda \) be an eigenvalue and \( v \) an associated eigenvector. Then \( (A - \lambda I)v = 0 \), so
\[
e^{tA}v = e^{t\lambda} \cdot e^{t(A - \lambda I)}v \\
= e^{t\lambda} \left[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots\right]v \\
= e^{t\lambda} \left[v + t(A - \lambda I)v + \frac{t^2}{2!}(A - \lambda I)^2v + \cdots\right] \\
= e^{t\lambda}v
\]
• The infinite series truncates, so we can compute \( e^{tA}v \).
Matrices with One Eigenvalue

A has characteristic polynomial \( p(\lambda) = (\lambda - \lambda_1)^n \).

- Cayley-Hamilton Theorem: If \( p(\lambda) \) is the characteristic polynomial of the matrix \( A \) then \( p(A) = 0I \).
- In our case \( (A - \lambda_1 I)^n = 0I \), so

\[
e^{tA} = e^{\lambda_1 t} \cdot [I + t(A - \lambda_1 I) + \frac{t^2}{2!}(A - \lambda_1 I)^2 + \cdots + \frac{t^{n-1}}{(n-1)!}(A - \lambda_1 I)^{n-1}]\]

Example 3

\[
A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix}
\]

- \( p(\lambda) = (\lambda + 2)^2 \).

\[
A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (A + 2I)^2 = 0I
\]

\[
e^{tA} = e^{-2t} \begin{pmatrix} 1 - t & t \\ -t & 1 + t \end{pmatrix}.
\]

Example 4

\[
A = \begin{pmatrix} 0 & -9 & 27 \\ -2 & 3 & -18 \\ -1 & 3 & -12 \end{pmatrix}
\]

- \( p(\lambda) = (\lambda + 3)^3 \). \( (A + 3I)^2 = 0I \).

\[
e^{tA} = e^{-3t} \begin{pmatrix} 1 + 3t & -9t & 27t \\ -2t & 1 + 6t & -18t \\ -t & 3t & 1 - 9t \end{pmatrix}.
\]
Example 2, Reprise

- Distinct eigenvalues $\lambda_1 = -3$ & $\lambda_2 = -1$.
- Different from previous two examples.
- $\lambda_1 = -3$ has algebraic multiplicity 1, and geometric multiplicity 1. So there is one exponential solution
  
  \[ x_1(t) = e^{\lambda_1 t}v_1 = e^{-3t}(-1/2, 3/2, 1)^T. \]

- $\lambda_2 = -1$ has algebraic multiplicity 2, and geometric multiplicity 1. So there is only one exponential solution
  
  \[ x_2(t) = e^{\lambda_2 t}v_2 = e^{-t}(-1/2, 1, 1)^T. \]

- However, $\text{null}((A - \lambda_2 I)^2)$ has dimension 2, with basis $(0, 1, 1)^T$ and $(1, 0, 0)^T$.
- $v_2$ is in $\text{null}((A - \lambda_2 I)^2)$
- Set $v_3 = (1, 0, 0)^T$.

- If $v \in \text{null}((A - \lambda_2 I)^2)$ then
  
  \[ e^{tA}v = e^{\lambda_2 t}[I + t(A - \lambda_2 I) + \frac{t^2}{2!}(A - \lambda_2 I)^2 + \cdots]v = e^{\lambda_2 t}[v + t(A - \lambda_2 I)v]. \]

- Using $v_3$ we get the third solution
  
  \[ x_3(t) = e^{tA}v_3 = e^{-t}[v_3 + t(A + I)v_3] = e^{-t}(1 + 2t, -4t, -4t)^T. \]

- $x_1$, $x_2$, and $x_3$ are a fundamental set of solutions.
Summary

• In Examples 3 & 4 the matrix has one eigenvalue.
• The series for $e^{t(A-\lambda I)}$ truncates to a finite sum.
• In Example 2 the matrix had two eigenvalues.
• The series for $e^{t(A-\lambda I)}$ does not truncate for any $\lambda$.
• However, the series for $e^{t(A-\lambda I)^2}v$ does truncate if $(A-\lambda^2I)^2v = 0$.

Generalized Eigenvectors

Definition: If $\lambda$ is an eigenvalue of $A$ and $(A-\lambda I)^p v = 0$ for some integer $p \geq 1$, then $v$ is called a generalized eigenvector associated with $\lambda$.
• The series for $e^{t(A-\lambda I)}v$ truncates to a finite sum if $v$ is a generalized eigenvector associated with $\lambda$.
• We can compute $e^{tA}v$.

Theorem: If $\lambda$ is an eigenvalue of $A$ with algebraic multiplicity $q$, then there is an integer $p \leq q$ such that $\text{null}((A-\lambda I)^p)$ has dimension $q$.
• For each generalized eigenvector $v$ we can compute $e^{tA}v$.
• We can find $q$ linearly independent solutions associated with the eigenvalue $\lambda$. 
Procedure for $\lambda$ of algebraic multiplicity $q$

To find $q$ linearly independent solutions associated with $\lambda$:
- Find the smallest integer $p$ such that $\text{null}((A - \lambda I)^p)$ has dimension $q$.
- Find a basis $v_1, v_2, \ldots, v_q$ of $\text{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$
  \[ x_j(t) = e^{\lambda t}v_j = e^{\lambda t}v_j + t(A - \lambda I)v_j + \frac{t^2}{2!}(A - \lambda I)^2v_j + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}v_j \]

Example

- Use MATLAB.

Procedure for a Complex Eigenvalue

If $\lambda$ is complex of algebraic multiplicity $q$, then $\lambda$ also has multiplicity $q$.
- Find the smallest integer $p$ such that $\text{null}((A - \lambda I)^p)$ has dimension $q$.
- Find a basis $w_1, w_2, \ldots, w_q$ of $\text{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$
  \[ x_j(t) = e^{A t}w_j. \]
• For \( j = 1, 2, \ldots, q \)

\[
\begin{align*}
    x_j(t) &= e^{\lambda t}w_j + t(A - \lambda I)w_j \\
           &\quad + \frac{t^2}{2!}(A - \lambda I)^2w_j + \cdots \\
           &\quad + \frac{(p-1)!}{(p-1)!}(A - \lambda I)^{p-1}w_j
\end{align*}
\]

• For \( j = 1, 2, \ldots, q \) set

\[
\begin{align*}
    x_j(t) &= \text{Re}(z_j(t)) \quad \text{and} \\
    y_j(t) &= \text{Im}(z_j(t)).
\end{align*}
\]