Math 211
Lecture #31
Stability of Solutions
Higher Order Equations

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Exponential of a Matrix
Definition: The exponential of the $n \times n$ matrix $A$ is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!} A^n.$$ 

Theorem: The solution to the initial value problem

$$x' = Ax \quad \text{with} \quad x(0) = v$$ 

is $x(t) = e^{tA}v$.

Computing $e^{tA}v$

- If $\lambda$ is an eigenvalue and $v$ is an associated eigenvector, then $e^{tA}v = e^{t\lambda}v$.
- If $(A - \lambda I)^p v = 0$ for some integer $p \geq 1$, then

$$e^{tA}v = e^{t\lambda} \left[ v + t(A - \lambda) v + \frac{t^2}{2!} (A - \lambda)^2 v + \cdots + \frac{t^{p-1}}{(p-1)!} (A - \lambda)^{p-1} v \right].$$
Generalized Eigenvectors

Definition: If \( \lambda \) is an eigenvalue of \( A \) and 
\[
(A - \lambda I)^p v = 0
\]
for some integer \( p \geq 1 \), then \( v \) is called a generalized eigenvector associated with \( \lambda \).

- We can compute \( e^{tA}v \) for all such \( v \).

Theorem: If \( \lambda \) is an eigenvalue of \( A \) with algebraic multiplicity \( q \), then there is an integer \( p \leq q \) such that 
\[
\text{null}((A - \lambda I)^p)
\]
has dimension \( q \).

- We can find \( q \) linearly independent solutions associated with the eigenvalue \( \lambda \).

Procedure for \( \lambda \) of algebraic multiplicity \( q \)

To find \( q \) linearly independent solutions associated with \( \lambda \):
- Find the smallest integer \( p \) such that 
\[
\text{null}((A - \lambda I)^p)
\]
has dimension \( q \).
- Find a basis \( v_1, v_2, \ldots, v_q \) of \( \text{null}((A - \lambda I)^p) \).
- For \( j = 1, 2, \ldots, q \)
\[
x_j(t) = e^{tA}v_j
= e^{tA} \left[ v_j + t(A - \lambda I)v_j + \frac{t^2}{2!}(A - \lambda I)^2v_j + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}v_j \right]
\]

Example

- Use MATLAB.
Procedure for a Complex Eigenvalue

If $\lambda$ is a complex eigenvalue of algebraic multiplicity $q$, then $\lambda$ also has algebraic multiplicity $q$.

- Find the smallest integer $p$ such that $\text{null}((A - \lambda I)^p)$ has dimension $q$.
- Find a basis $w_1, w_2, \ldots, w_q$ of $\text{null}((A - \lambda I)^p)$.

- For $j = 1, 2, \ldots, q$ we have solutions
  \[ z_j(t) = e^{\lambda t}w_j = e^{\lambda t} \left[ w_j + t(A - \lambda I)w_j + \frac{t^2}{2!}(A - \lambda I)^2w_j + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}w_j \right] \]
- $x_1, \ldots, x_q$ together with $\pi_1, \ldots, \pi_q$ are $2q$ linearly independent complex valued solutions.
- For $j = 1, 2, \ldots, q$ set $x_j(t) = \text{Re}(z_j(t))$ and $y_j(t) = \text{Im}(z_j(t))$. These are $2q$ linearly independent real valued solutions.

Stability

Autonomous system $x' = f(x)$ with an equilibrium point at $x_0$.

- Basic question: What happens to all solutions as $t \to \infty$?
- $x_0$ is stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that a solution $x(t)$ with $|x(0) - x_0| < \delta$ implies $|x(t) - x_0| < \epsilon$ for all $t \geq 0$.
- Every solution that starts close to $x_0$ stays close to $x_0$. 
• $x_0$ is asymptotically stable if it is stable and there is an $\eta > 0$ such that if $x(t)$ is a solution with $|x(0) - x_0| < \eta$, then $x(t) \to x_0$ as $t \to \infty$.
  - $x_0$ is called a sink.
  - Every solution that starts close to $x_0$ approaches $x_0$.
• $x_0$ is unstable if there is an $\epsilon > 0$ such that for any $\delta > 0$ there is a solution $x(t)$ with $|x(0) - x_0| < \delta$ with the property that there are values of $t > 0$ such that $|x(t) - x_0| > \epsilon$.
  - There are solutions starting arbitrarily close to $x_0$ that move away from $x_0$.

Examples $D = 2$
• Sinks are asymptotically stable.
  - The eigenvalues have negative real part.
• Sources are unstable.
  - The eigenvalues have positive real part.
• Saddles are unstable.
  - One eigenvalue has positive real part.
• Centers are stable but not asymptotically stable.
  - The eigenvalues have real part $= 0$.

Theorem: Let $A$ be an $n \times n$ real matrix.
• Suppose the real part of every eigenvalue of $A$ is negative. Then 0 is an asymptotically stable equilibrium point for the system $x' = Ax$.
• Suppose $A$ has at least one eigenvalue with positive real part. Then 0 is an unstable equilibrium point for the system $x' = Ax$. 
Examples

- $D = 2$
- $T^2 - 4D = 0.$
  - $T < 0 \Rightarrow$ sink. $T > 0 \Rightarrow$ source.
- $y' = Ay,$
  
  $A = \begin{pmatrix}
  -2 & -18 & -7 & -14 \\
  1 & 6 & 2 & 5 \\
  2 & 2 & -3 & 0 \\
  -2 & -8 & -1 & -6
  \end{pmatrix}.$

- $A$ has eigenvalues $-1, -2, & -1 \pm i.$
- $0$ is asymptotically stable.

Higher Order Equations

$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_n y = 0$

- Second order: $y'' + py' + qy = 0.$
- Equivalent system: $x' = Ax,$ where
  
  $x = \begin{pmatrix} y \\ y' \end{pmatrix}$ and $A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$

- A fundamental set of solutions for the system consists of two linearly independent solutions.

Linear Independence

Definition: Two functions $u(t)$ and $v(t)$ are linearly independent if neither is a constant multiple of the other.

- $u(t)$ and $v(t)$ are linearly independent solutions to $y'' + py' + qy = 0 \Rightarrow (u'') \& (v'')$ are linearly independent solutions to the equivalent system.
General Solution

Theorem: Suppose that $y_1(t)$ & $y_2(t)$ are linearly independent solutions to the equation

$$y'' + py' + qy = 0.$$ 

Then the general solution is

$$y(t) = C_1 y_1(t) + C_2 y_2(t).$$

Definition: A set of two linearly independent solutions is called a fundamental set of solutions.

Solutions to $y'' + py' + qy = 0$.

- Equivalent system: $x' = Ax$, where 
  $$x = \begin{pmatrix} y \\ y' \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 0 & 1 \\ -q & -p \end{pmatrix}.$$ 

- Look for exponential solutions $y(t) = e^{\lambda t}$.

- Characteristic equation: $\lambda^2 + p\lambda + q = 0$.

- Characteristic polynomial: $\lambda^2 + p\lambda + q$.

- Same for the 2nd order equation and the system.

Real Roots

- If $\lambda$ is a root to the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.

- If $\lambda$ is a root to the characteristic polynomial of multiplicity 2, then $y_1(t) = e^{\lambda t}$ and $y_2(t) = te^{\lambda t}$ are linearly independent solutions.
Complex Roots

- If \( \lambda = \alpha + i\beta \) is a complex root of the characteristic equation, then so is \( \bar{\lambda} = \alpha - i\beta \).
- A complex valued fundamental set of solutions is
  \[ z(t) = e^{\lambda t} \quad \text{and} \quad \bar{z}(t) = e^{\bar{\lambda} t}. \]
- A real valued fundamental set of solutions is
  \[ x(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = e^{\alpha t} \sin \beta t. \]

Examples

- \( y'' - 5y' + 6y = 0 \).
- \( y'' + 25y = 0 \).
- \( y'' + 4y' + 13y = 0 \).

Key to Computing \( e^{tA} \) or \( e^{tA}v \)

Suppose that \( A \) a \( n \times n \) matrix, and \( \lambda \) a number (an eigenvalue).

- \( A = \lambda I + (A - \lambda I); \{ \lambda I \ & A - \lambda I \ \text{commute.} \} \)
  \[
e^{tA} = e^{t(\lambda I + (A - \lambda I))} \\
  = e^{t\lambda I} \cdot e^{t(A - \lambda I)} \\
  = e^{t\lambda} \cdot e^{t(A - \lambda I)} \\
  = e^{t\lambda} \cdot \left[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots \right]
\]
Multiplicities

A an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is
  $$p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}.$$  
- The algebraic multiplicity of $\lambda_j$ is $q_j$.
- The geometric multiplicity of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$. 