Forced, Damped Harmonic Motion

\[ x'' + 2cx' + \omega_0^2 x = A \cos \omega t \]

- Characteristic polynomial: \( P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2 \)
- General Solution
  \[ x(t) = G(\omega)A \cos(\omega t - \phi) + x_h(t). \]
  - Transient term \( x_h(t) \) dies out exponentially.
  - Steady-state solution \( x_p(t) = G(\omega)A \cos(\omega t - \phi) \).
  - Gain: \( G(\omega) = 1/\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2}\omega^2} \).
  - Phase: \( \phi = \arccot \left( (\omega_0^2 - \omega^2)/2c\omega \right) \).

Steady-State Solution

\[ x_p(t) = G(\omega)A \cos(\omega t - \phi). \]

- The forcing function is \( A \cos \omega t \).
- The steady-state response is oscillatory.
  - The amplitude is \( G(\omega) \) times the amplitude of the forcing term.
  - The steady-state oscillation is at the forcing frequency.
  - There is a phase shift of \( \phi/\omega \).
Interacting Species

- Two species with populations $x_1$ & $x_2$.
- Interaction between the species can be helpful or detrimental.
- Basic model
  \[
  \begin{align*}
  x_1' &= r_1 x_1 \\
  x_2' &= r_2 x_2
  \end{align*}
  \]
- $r_1$ & $r_2$ are the reproductive rates.

Reproductive Rates

- If $x_2 = 0$ the reproductive rate for $x_1$ is
  \[
  r_1 = a_1 - b_1 x_1.
  \]
- $a_1 > 0 \Rightarrow$ natural growth.
- $a_1 < 0 \Rightarrow$ natural decline.
- $b_1 = 0$ Malthusian growth.
- $b_1 > 0$ logistic growth.

- If $x_2 > 0$ the reproductive rate for $x_1$ is
  \[
  r_1 = a_1 - b_1 x_1 + c_1 x_2.
  \]
- $c_1 > 0 \Rightarrow$ interaction is helpful to $x_1$.
- $c_1 < 0 \Rightarrow$ interaction is detrimental to $x_1$.
- The reproductive rate for $x_2$ is
  \[
  r_2 = a_2 - b_2 x_2 + c_2 x_1.
  \]
- The model for interacting species is
  \[
  \begin{align*}
  x_1' &= (a_1 - b_1 x_1 + c_1 x_2) x_1 \\
  x_2' &= (a_2 - b_2 x_2 + c_2 x_1) x_2
  \end{align*}
  \]
Predator Prey Model
Rabbits & foxes, fish & sharks, and cottony cushion scale insect & ladybird beetle.
• $F = \text{fish} \ & \ S = \text{sharks}$.
  \[
  F' = (a - bS)F \\
  S' = (-c + dF)S
  \]
  or
  \[
  F' = (a - eF - bS)F \\
  S' = (-c + dF)S
  \]
  \[a = 3, \ b = 3, \ c = 1, \ d = 3, \ e = 3.\]

Competing Species
Cattle and sheep.
• $x_1$ and $x_2$ competing for resources.
  \[
  x'_1 = (a_1 - b_1x_1 + c_1x_2)x_1 \\
  x'_2 = (a_2 - b_2x_2 + c_2x_1)x_2
  \]
  • $a_2 > 0, \ b_1 > 0, \ & \ c_1 < 0$
  • Example:
    \[
    x' = (5 - 2x - y)x \\
    y' = (7 - 2x - 3y)y
    \]

Linearization
The principal idea of differential calculus:
• Approximate nonlinear mathematical objects by linear ones.
• Example: Approximate the function $f(y)$ near $y_0$ by linear function.
  \[
  f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)
  \]
  where \( \lim_{h \to 0} \frac{R(h)}{h} = 0 \).
  • The linear function is $L(h) = f(y_0) + f'(y_0)h$. 
Linearization of an ODE

\[ y' = f(y) \]

- Assume \( f(y_0) = 0 \) and \( f'(y_0) \neq 0 \).
- Set \( y = y_0 + u \). Get
  \[ u' = f(y_0 + u) = f'(y_0)u + R(u) \]

- Approximate by the linear differential equation
  \[ \tilde{u}' = f'(y_0)\tilde{u} \]

- If \( f'(y_0) \neq 0 \) the equilibrium point of the linearization at 0 has the same stability properties as that of the nonlinear equation at \( y_0 \).
  - \( f'(y_0) > 0 \Rightarrow y_0 \) is unstable.
  - \( f'(y_0) < 0 \Rightarrow y_0 \) is asymptotically stable.
- We can solve the linearization explicitly.

Linearization of a Planar System

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

- Assume \( (x_0, y_0) \) is an equilibrium point, so
  \[ f(x_0, y_0) = g(x_0, y_0) = 0 \]
We have by Taylor’s theorem
\[ f(x_0 + u, y_0 + v) = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v) \]
\[ g(x_0 + u, y_0 + v) = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v) \]
where \( R_f(u, v) \to 0 \) and \( R_g(u, v) \to 0 \)

- Set \( x = x_0 + u \) and \( y = y_0 + v \). The system becomes
  \[ u' = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v) \]
  \[ v' = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v) \]

Linearization at \( (x_0, y_0) \)

\[ \tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v} \]
\[ \tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v} \]

- This is a linear system.
  - We can solve it explicitly.
  - Does it give information about the original nonlinear system?
Matrix Form of the Linearization

Set \( \mathbf{u} = (\tilde{u}, \tilde{v})^T \) and introduce the Jacobian matrix

\[
\mathbf{J} = \begin{pmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
\]

- The linearization becomes \( \mathbf{u}' = \mathbf{J}\mathbf{u} \).

Theorem: Consider the planar system

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

where \( f \) and \( g \) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
- All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - Center and others. Occupy pieces of the boundaries.
Examples

- Predator prey
- Competing species
- Center

\[ \begin{align*}
  x' &= y + \alpha x(x^2 + y^2) \\
  y' &= -x + \alpha y(x^2 + y^2)
\end{align*} \]