Math 211

Lecture #36

Forced Harmonic Motion
Nonlinear Systems

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Forced, Damped Harmonic Motion

\[ x'' + 2cx' + \omega_0^2 x = A \cos \omega t \]

- **Ch. polynomial:** \( P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2 \)
- **General Solution**
  \[ x(t) = G(\omega)A \cos(\omega t - \phi) + x_h(t). \]

- **Transient term** \( x_h(t) \) dies out exponentially.
- **Steady-state solution** \( x_p(t) = G(\omega)A \cos(\omega t - \phi). \)
  
  - **Gain:** \( G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}. \)
  - **Phase:** \( \phi = \arccot \left( \frac{(\omega_0^2 - \omega^2)}{2c\omega} \right). \)
Steady-State Solution

\[ x_p(t) = G(\omega)A \cos(\omega t - \phi). \]

- The forcing function is \( A \cos \omega t \).
- The steady-state response is oscillatory.
  - The amplitude is \( G(\omega) \) times the amplitude of the forcing term.
  - The steady-state oscillation is at the forcing frequency.
  - There is a phase shift of \( \phi/\omega \).
Interacting Species

- Two species with populations $x_1$ & $x_2$.
- Interaction between the species can be helpful or detrimental.
- Basic model

\[
x'_1 = r_1 x_1 \\
x'_2 = r_2 x_2
\]

- $r_1$ & $r_2$ are the \textit{reproductive rates}. 
Reproductive Rates

- If $x_2 = 0$ the reproductive rate for $x_1$ is
  \[ r_1 = a_1 - b_1 x_1. \]

  - $a_1 > 0 \Rightarrow$ natural growth.
  - $a_1 < 0 \Rightarrow$ natural decline.
  - $b_1 = 0$ Malthusian growth.
  - $b_1 > 0$ logistic growth.
• If \( x_2 > 0 \) the **reproductive rate** for \( x_1 \) is

\[
 r_1 = a_1 - b_1 x_1 + c_1 x_2.
\]

• \( c_1 > 0 \Rightarrow \) interaction is helpful to \( x_1 \).

• \( c_1 < 0 \Rightarrow \) interaction is detrimental to \( x_1 \).

• The reproductive rate for \( x_2 \) is

\[
 r_2 = a_2 - b_2 x_2 + c_2 x_1.
\]

• The model for **interacting species** is

\[
 x_1' = (a_1 - b_1 x_1 + c_1 x_2)x_1 \\
 x_2' = (a_2 - b_2 x_2 + c_2 x_1)x_2
\]
Predator Prey Model

Rabbits & foxes, fish & sharks, and cottony cushion scale insect & ladybird beetle.

- $F = \text{fish}$ & $S = \text{sharks}$.

$$F' = (a - bS)F$$
$$S' = (-c + dF)S$$

or

$$F' = (a - eF - bS)F$$
$$S' = (-c + dF)S$$

$a = 3$, $b = 3$, $c = 1$, $d = 3$, $e = 3$. 

Competing Species

Cattle and sheep.

- $x_1$ and $x_2$ competing for resources.

\[
x_1' = (a_1 - b_1 x_1 + c_1 x_2)x_1
\]
\[
x_2' = (a_2 - b_2 x_2 + c_2 x_1)x_2
\]

- $a_i > 0$, $b_i > 0$, & $c_i < 0$

- Example:

\[
x' = (5 - 2x - y)x
\]
\[
y' = (7 - 2x - 3y)y
\]
Linearization

The principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.

- Example: Approximate the function $f(y)$ near $y_0$ by a linear function.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

where $$\lim_{h \to 0} \frac{R(h)}{h} = 0.$$

- The linear function is $L(h) = f(y_0) + f'(y_0)h$. 
Linearization of an ODE

\[ y' = f(y) \]

- Assume \( f(y_0) = 0 \) and \( f'(y_0) \neq 0 \).
- Set \( y = y_0 + u \). Get

\[ u' = f(y_0 + u) \]
\[ = f'(y_0)u + R(u) \]

- Approximate by the linear differential equation

\[ \tilde{u}' = f'(y_0)\tilde{u} \]
• If $f'(y_0) \neq 0$ the equilibrium point of the linearization at 0 has the same stability properties as that of the nonlinear equation at $y_0$.
  
  ♦ $f'(y_0) > 0 \Rightarrow y_0$ is unstable.
  
  ♦ $f'(y_0) < 0 \Rightarrow y_0$ is asymptotically stable.

• We can solve the linearization explicitly.
Linearization of a Planar System

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

- Assume \((x_0, y_0)\) is an equilibrium point, so
\[ f(x_0, y_0) = g(x_0, y_0) = 0 \]
We have by Taylor’s theorem

\[ f(x_0 + u, y_0 + v) = \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v) \]

\[ g(x_0 + u, y_0 + v) = \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v) \]

where \( \frac{R_f(u, v)}{\sqrt{u^2 + v^2}} \to 0 \) and \( \frac{R_g(u, v)}{\sqrt{u^2 + v^2}} \to 0 \)
● Set \( x = x_0 + u \) and \( y = y_0 + v \). The system becomes

\[
\begin{align*}
    u' &= \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v) \\
    v' &= \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)
\end{align*}
\]
Linearization at \((x_0, y_0)\)

\[
\begin{align*}
\tilde{u}' &= \frac{\partial f}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0)\tilde{v} \\
\tilde{v}' &= \frac{\partial g}{\partial x}(x_0, y_0)\tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0)\tilde{v}
\end{align*}
\]

- This is a linear system.
  - We can solve it explicitly.
  - Does it give information about the original nonlinear system?
Matrix Form of the Linearization

Set $u = (\tilde{u}, \tilde{v})^T$ and introduce the $Jacobian matrix$

$$J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix}$$

• The linearization becomes

$$u' = Ju.$$
**Theorem:** Consider the planar system

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

where \( f \) and \( g \) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.
Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.

- Nongeneric types
  - Center and others. Occupy pieces of the boundaries.
Examples

- Predator prey
- Competing species
- Center

\[ x' = y + \alpha x(x^2 + y^2) \]
\[ y' = -x + \alpha y(x^2 + y^2) \]