Math 211

Lecture #37

Linearization

November 26, 2001
The principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Example: Approximate the function $f(y)$ near $y_0$ by a linear function.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

where $\lim_{h \to 0} \frac{R(h)}{h} = 0$.

- The linear function is $L(h) = f(y_0) + f'(y_0)h$. 

Linearization
Linearization of an ODE

\[ y' = f(y) \]

- Assume \( f(y_0) = 0 \) and \( f'(y_0) \neq 0 \).
- Set \( y = y_0 + u \). Get

\[ u' = f(y_0 + u) = f'(y_0)u + R(u) \]

- Approximate by the linear differential equation

\[ \tilde{u}' = f'(y_0)\tilde{u} \]
If $f'(y_0) \neq 0$ the equilibrium point of the linearization at 0 has the same stability properties as that of the nonlinear equation at $y_0$.

- $f'(y_0) > 0 \Rightarrow y_0$ is unstable.
- $f'(y_0) < 0 \Rightarrow y_0$ is asymptotically stable.

- We can solve the linearization explicitly.
Linearization of a Planar System

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

- \((x_0, y_0)\) is an equilibrium point so

\[ f(x_0, y_0) = g(x_0, y_0) = 0 \]
We have by Taylor’s theorem

\[ f(x_0 + u, y_0 + v) = \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v) \]

\[ g(x_0 + u, y_0 + v) = \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v) \]

where \( \frac{R_f(u, v)}{\sqrt{u^2 + v^2}} \to 0 \) and \( \frac{R_g(u, v)}{\sqrt{u^2 + v^2}} \to 0 \)
Set \( x = x_0 + u \) and \( y = y_0 + v \). The system becomes

\[
\begin{align*}
    u' &= \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v) \\
    v' &= \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v)
\end{align*}
\]
Linearization at \((x_0, y_0)\)

\[
\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v}
\]

\[
\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v}
\]

- This is a linear system.
  - It is called the linearization of the system at the equilibrium point \((x_0, y_0)\).
  - We can solve it explicitly.
  - Does it give information about the original system?
Matrix Form of the Linearization

Set $\mathbf{u} = (\tilde{u}, \tilde{v})^T$ and introduce the *Jacobian matrix*

$$
J = \begin{pmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
$$

- The linearization becomes

$$
\mathbf{u}' = J\mathbf{u}.
$$

- The behavior of solutions to the linearization is determined by the eigenvalues of the Jacobian.
Theorem: Consider the planar system

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

where \( f \) and \( g \) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.
Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.

- Nongeneric types
  - Center and others, which occupy pieces of the boundaries between the generic points.
Examples

- Predator prey

\[ F' = (3 - 3S)F \]
\[ S' = (-1 + 3F)S \]

or

\[ F' = (3 - 3F - 3S)F \]
\[ S' = (-1 + 3F)S \]
• Competing species

\[ x' = (5 - 2x - y)x \]
\[ y' = (7 - 2x - 3y)y \]

• Center

\[ x' = y + \alpha x(x^2 + y^2) \]
\[ y' = -x + \alpha y(x^2 + y^2) \]

♦ \( \alpha > 0 \Rightarrow (0, 0)^T \) is unstable.

♦ \( \alpha < 0 \Rightarrow (0, 0)^T \) is a sink.