Math 211

Lecture #38
Linearization in Higher Dimension

November 28, 2001

Linearization of a Planar System

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

- \((x_0, y_0)\) is an eq. point so \(f(x_0, y_0) = g(x_0, y_0) = 0\).
- Linearization at \((x_0, y_0)\)
  \[
  \begin{align*}
  \tilde{u}' &= \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v} \\
  \tilde{v}' &= \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v}
  \end{align*}
  \]

Matrix Form of the Linearization

Set \(u = (\tilde{u}, \tilde{v})^T\) and introduce the Jacobian matrix

\[
J = \begin{bmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{bmatrix}
\]

- The linearization becomes
  \[
  u' = J u.
  \]
Theorem: Consider the planar system
\[ \begin{align*}
    x' &= f(x, y) \\
y' &= g(x, y)
\end{align*} \]
where \( f \) and \( g \) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.

Generic Equilibrium Points
- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
- Occupy large open subsets of the trace-determinant plane.
- Nongeneric types: center and others.
- Occupy pieces of the boundaries between the generic points.
- Example:
\[ \begin{align*}
x' &= y + \alpha x(x^2 + y^2) \\
y' &= -x + \alpha y(x^2 + y^2)
\end{align*} \]

Higher Dimensional Systems
Autonomous equation \( y' = f(y) \)
- \( y = (y_1, y_2, \cdots, y_n)^T \)
- \( f(y) = (f_1(y), f_2(y), \cdots, f_n(y))^T \)
- \( J \) is the Jacobian matrix
\[ f(y_0 + u) = J(y_0)u + R(u) \quad \text{where} \quad \lim_{u \to 0} \frac{R(u)}{|u|} = 0. \]
- Set \( y = y_0 + u \). The system becomes
\[ u' = J(y_0)u + R(u). \]
- The linearization is \( u' = J(y_0)u \).
Theorem: Suppose that \( y_0 \) is an equilibrium point for \( y' = f(y) \). Let \( J \) be the Jacobian of \( f \) at \( y_0 \).

1. Suppose that the real part of every eigenvalue of \( J \) is negative. Then \( y_0 \) is an asymptotically stable equilibrium point.
2. Suppose that \( J \) has at least one eigenvalue with positive real part. Then \( y_0 \) is an unstable equilibrium point.

Example

\[
\begin{align*}
x' &= -2x - 4y + 2xy \\
y' &= x - 6y + x^2 - y^2
\end{align*}
\]

- The origin \((0, 0)\) is an equilibrium point.
- The Jacobian has one eigenvalue, \( \lambda = -4 \), of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem ⇒ the origin is a sink.

The Lorenz System

\[
\begin{align*}
x' &= -ax + ay \\
y' &= rx - y - xz \\
z' &= -bz + xy
\end{align*}
\]

- Equilibrium points.
  - \((r \leq 1)\) \((0, 0, 0)\)
  - \((r > 1)\) Set \( s = \sqrt{b(r - 1)} \). The equilibrium points are \((0, 0, 0)\), and \( c^\pm = (\pm s, \pm s, r - 1) \).
• The Jacobian is
\[ J = \begin{pmatrix} -a & a & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix} \]

• Use \( a = 10 \) and \( b = 8/3 \).
• \((0,0,0)\)
  ▶ If \( r < 1 \) \((0,0,0)\) is asymptotically stable.
  ▶ If \( r > 1 \) \((0,0,0)\) is unstable.

• \( c^+ \) and \( c^- \)
  ▶ For \( 1 < r < 470/19 \approx 24.74 \), \( c^+ \) and \( c^- \) are asymptotically stable.
  ▶ For \( r > 470/19 \approx 24.74 \), \( c^+ \) and \( c^- \) are unstable.

• As \( r \) varies the Lorenz system displays a wide variety of behaviors.
  ▶ For \( r = 28 \) we have Lorenz’s strange attractor.
  ▶ For \( r = 100 \) there is a periodic attractor.
  ▶ For \( r = 200 \) there is another strange attractor.
Invariant Sets

Definition: A set $S$ is (positively) invariant for the system $\dot{y} = f(y)$ if $y(0) = y_0 \in S$ implies that $y(t) \in S$ for all $t \geq 0$.

- Examples:
  - An equilibrium point.
  - Any solution curve.

Example — Competing Species

\[
\begin{align*}
x' &= (5 - 2x - y)x \\
y' &= (7 - 2x - 3y)y
\end{align*}
\]

- The positive $x$- and $y$-axes are invariant.
- The positive quadrant is invariant.
- Populations should remain nonnegative.
- The set $S = \{(x, y) | 0 < x < 3, 0 < y < 3\}$ is positively invariant.

Nullclines

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

Definition: The $x$-nullcline is the set defined by $f(x, y) = 0$. The $y$-nullcline is the set defined by $g(x, y) = 0$.

- Along the $x$-nullcline the vector field points up or down.
- Along the $y$-nullcline the vector field points left or right.
Competing Species

\[ x' = (5 - 2x - y)x \]
\[ y' = (7 - 2x - 3y)y \]

- The \( x \)-nullcline consists of the two lines \( x = 0 \) and \( 2x + y = 5 \).
- The \( y \)-nullcline consists of the two lines \( y = 0 \) and \( 2x + 3y = 7 \).
- The nullclines intersect at the equilibrium points.

Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.

This information allows us to predict that all solutions in the positive quadrant \(\to (2, 1)\) as \( t \to \infty \).

Competing Species – 2nd Example

\[ x' = (1 - x - y)x \]
\[ y' = (4 - 7x - 3y)y \]

- The equilibrium point at \((1/4, 3/4)\) is a saddle point.
- All solutions go to either \((0, 4/3)\) or \((1, 0)\).
Definition: The basin of attraction of a sink \( y_0 \) consists of all points \( y \) such that the solution starting at \( y \) approaches \( y_0 \) as \( t \to \infty \).

- In the example, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called separatrices. (Separatrices is the plural of separatrix.)