Math 211

Lecture #39

Invariant Sets

November 30, 2001

Review of Methods

Linearization at an equilibrium point

- \( y' = f(y) \) has an equilibrium point at \( y_0 \).
- The linearization \( u' = J(y_0)u \) has an equilibrium point at \( u = 0 \).
- The linearization can sometimes predict the behavior of solutions to the nonlinear system near the equilibrium point.
- The linearization gives only local information.

Theorem: Consider the planar system

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

where \( f \) and \( g \) are continuously differentiable. Suppose that \( (x_0, y_0) \) is an equilibrium point. If the linearization at \( (x_0, y_0) \) has a generic equilibrium point at the origin, then the equilibrium point at \( (x_0, y_0) \) is of the same type.
Theorem: Suppose that $y_0$ is an equilibrium point for $y' = f(y)$. Let $J$ be the Jacobian of $f$ at $y_0$.

1. Suppose that the real part of every eigenvalue of $J$ is negative. Then $y_0$ is an asymptotically stable equilibrium point.

2. Suppose that $J$ has at least one eigenvalue with positive real part. Then $y_0$ is an unstable equilibrium point.

Invariant Sets

Definition: A set $S$ is (positively) invariant for the system $y' = f(y)$ if $y(0) = y_0 \in S$ implies that $y(t) \in S$ for all $t \geq 0$.

- Examples:
  - An equilibrium point.
  - Any solution curve.

Example — Competing Species

$$x' = (5 - 2x - y)x$$
$$y' = (7 - 2x - 3y)y$$

- The positive $x$- and $y$-axes are invariant.
- The positive quadrant is invariant.
- Populations should remain nonnegative.
- The set $S = \{(x, y) | 0 < x < 3, \ 0 < y < 3 \}$ is positively invariant.
Nullclines

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

**Definition:** The \( x \)-nullcline is the set defined by \( f(x, y) = 0 \). The \( y \)-nullcline is the set defined by \( g(x, y) = 0 \).

- Along the \( x \)-nullcline the vector field points up or down.
- Along the \( y \)-nullcline the vector field points left or right.

Competing Species

\[ x' = (5 - 2x - y)x \]
\[ y' = (7 - 2x - 3y)y \]

- The \( x \)-nullcline consists of the two lines \( x = 0 \) and \( 2x + y = 5 \).
- The \( y \)-nullcline consists of the two lines \( y = 0 \) and \( 2x + 3y = 7 \).
- The nullclines intersect at the equilibrium points.

- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant \( \rightarrow (2, 1) \) as \( t \rightarrow \infty \).
Competing Species – 2nd Example

\[ x' = (1 - x - y)x \]
\[ y' = (4 - 7x - 3y)y \]

- The axes are invariant. The positive quadrant is invariant.
- The equilibrium point at \((1/4, 3/4)\) is a saddle point.
- Almost all solutions go to one of the nodal sinks \((0, 4/3)\) or \((1, 0)\).

Definition: The basin of attraction of a sink \(y_0\) consists of all points \(y\) such that the solution starting at \(y\) approaches \(y_0\) as \(t \to \infty\).

- In the example, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called separatrices. (Separatrices is the plural of separatrix.)

Summary

- Sometimes the understanding of invariant sets can help us understand the long-term behavior of all solutions.
- Nullclines can sometimes help us find informative invariant sets.
- None of this helps us understand the predator-prey system.