Math 211

Lecture #39

Invariant Sets

November 30, 2001
Review of Methods

Linearization at an equilibrium point

• $y' = f(y)$ has an equilibrium point at $y_0$.

• The linearization $u' = J(y_0)u$ has an equilibrium point at $u = 0$.

• The linearization can sometimes predict the behavior of solutions to the nonlinear system *near the equilibrium point*.

• The linearization gives only local information.
**Theorem:** Consider the planar system

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

where \( f \) and \( g \) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.
**Theorem:** Suppose that $y_0$ is an equilibrium point for $y' = f(y)$. Let $J$ be the Jacobian of $f$ at $y_0$.

1. Suppose that the real part of every eigenvalue of $J$ is negative. Then $y_0$ is an asymptotically stable equilibrium point.

2. Suppose that $J$ has at least one eigenvalue with positive real part. Then $y_0$ is an unstable equilibrium point.
Invariant Sets

**Definition:** A set $S$ is *(positively) invariant* for the system $y' = f(y)$ if $y(0) = y_0 \in S$ implies that $y(t) \in S$ for all $t \geq 0$.

- **Examples:**
  - An equilibrium point.
  - Any solution curve.
Example — Competing Species

\[ x' = (5 - 2x - y)x \]
\[ y' = (7 - 2x - 3y)y \]

- The positive \( x \)- and \( y \)-axes are invariant.
- The positive quadrant is invariant.
  - Populations should remain nonnegative.
- The set \( S = \{(x, y) \mid 0 < x < 3, \; 0 < y < 3\} \) is positively invariant.
Nullclines

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

**Definition:** The *x-nullcline* is the set defined by 
\[ f(x, y) = 0 \]. The *y-nullcline* is the set defined by 
\[ g(x, y) = 0 \].

- Along the *x-nullcline* the vector field points up or down.
- Along the *y-nullcline* the vector field points left or right.
Competing Species

\[ x' = (5 - 2x - y)x \]
\[ y' = (7 - 2x - 3y)y \]

- The \( x \)-nullcline consists of the two lines \( x = 0 \) and \( 2x + y = 5 \).
- The \( y \)-nullcline consists of the two lines \( y = 0 \) and \( 2x + 3y = 7 \).
- The nullclines intersect at the equilibrium points.
• Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.

• This information allows us to predict that all solutions in the positive quadrant \( \rightarrow (2, 1) \) as \( t \rightarrow \infty \).
Competing Species – 2nd Example

\[ x' = (1 - x - y)x \]
\[ y' = (4 - 7x - 3y)y \]

• The axes are invariant. The positive quadrant is invariant.

• The equilibrium point at \((1/4, 3/4)\) is a saddle point.

• Almost all solutions go to one of the nodal sinks \((0, 4/3)\) or \((1, 0)\).
Definition: The *basin of attraction* of a sink $y_0$ consists of all points $y$ such that the solution starting at $y$ approaches $y_0$ as $t \to \infty$.

- In the example, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called *separatrices*. (Separatrices is the plural of separatrix.)
Summary

- Sometimes the understanding of invariant sets can help us understand the long term behavior of all solutions.
- Nullclines can sometimes help us find informative invariant sets.
- None of this helps us understand the predator-prey system.