Math 211

Lecture #40

Long Term Behavior of Planar Systems

December 3, 2001
Basic Question about a System $y' = f(y)$

- What happens to all solutions as $t \to \infty$?
- What are the possibilities as $t \to \infty$?
  - Is there a small list of all possibilities?
  - We need a definitive notion of what a “possibility” is.
Limit Sets

**Definition:** The (forward) limit set of the solution $y(t)$ that starts at $y_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(y_0)$.

- $x \in \omega(y_0)$ if there is a sequence $t_k \to \infty$ such that $y(t_k) \to x$.

- What kinds of sets can be limit sets?
  - The empty set.
  - Equilibrium points.
  - Periodic solution curves.

- Is there a small list of all possible limit sets?
Properties of Limit Sets

**Theorem:** Suppose that the system $y' = f(y)$ is defined in the set $U$.

1. If the solution curve starting at $y_0$ stays in a bounded subset of $U$, then the limit set $\omega(y_0)$ is not empty.

2. Any limit set is both positively and negatively invariant.
Example

\[ x' = 5y + x(9 - x^2 - y^2) \]

\[ y' = -5x + y(9 - x^2 - y^2) \]

- The origin is a spiral source.
- In polar coordinates the system is
  \[ r' = r(9 - r^2) \]
  \[ \theta' = -5 \]
- All solution curves approach the circle \[ x^2 + y^2 = 9. \]
  - The circle \[ x^2 + y^2 = 9 \] is a solution curve.
Limit Cycle

**Definition:** A limit cycle is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as $t \to \infty$, it is an attracting limit cycle. If they spiral into the limit cycle as $t \to -\infty$, it is a repelling limit cycle.

- In the example the circle $x^2 + y^2 = 9$ is a limit cycle.
Types of Limit Set

- A limit cycle is a new type of phenomenon.
- However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two types of non-empty limits sets.
  - An equilibrium point.
  - A closed solution curve.
Example

\[ x' = (y + x/5)(1 - x^2) \]
\[ y' = -x(1 - y^2) \]

- The lines \( x = \pm 1 \) and \( y = \pm 1 \) are invariant.
- The unit square is invariant.
- The corners of the unit square are saddle points.
  - The lines \( x = \pm 1 \) and \( y = \pm 1 \) are separatrices.
- The origin is a spiral source.
- The limit set of any solution that starts in the unit square is the boundary of the unit square.
Planar Graph

Definition: A planar graph is a collection of points, called vertices, and non-intersecting curves, called edges, which connect the vertices. If the edges each have a direction the graph is said to be directed.

- The boundary of the unit square in the example is a directed planar graph.
Theorem: If $S$ is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbb{R}^2$, then $S$ is one of the following:

- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the Poincaré-Bendixson alternatives.
Remarks

- These three are the only possibilities.
- The closed solution curve could be a limit cycle.
- If a vertex of a limiting planar graph is a generic equilibrium point, then it must be a saddle point. The edges connecting this point must be separatrices.
Poincaré-Bendixson Theorem

**Theorem:** Suppose that $R$ is a closed and bounded planar region that is positively invariant for a planar system. If $R$ contains no equilibrium points, then there is a closed solution curve in $R$.

- The theorem is true if the set $R$ is negatively invariant.
Examples

- # 1.

\[ x' = x + y - x(x^2 + 3y^2) \]
\[ y' = -x + y - 2y^3 \]

- The set \( \{(x, y) \mid 0.5 \leq x^2 + y^2 \leq 1\} \) is positively invariant.

- # 2. Rayleigh's example: \( z'' + \mu z'[\,(z')^2 - 1] + z = 0 \).