Math 211

Review for the Final Exam

December 8, 2002

The Final Exam

• The final will be comprehensive, covering material from the entire semester.
• The final will emphasize the material covered since the last exam.
• These slides will cover primarily the material covered since the last exam. They do not cover all of the material on the exam.
• Questions about any of the material of the course will be answered.

The Themes of the Course

• Modeling.
  • Population, finance, mixing, motion, vibrating spring, electrical circuits, ...
• Exact solutions.
  • Separable and linear equations in dimension 1.
  • Linear equations in higher dimension.
    ▶ Matrix algebra.
• Second order equations.
• Numerical solutions.
• Geometric analysis.
Solving $x' = Ax$

- $A$ is an $n \times n$ matrix.
- Solution strategy: Look for a fundamental set of solutions, i.e., $n$ linearly independent solutions.
- The function $x(t) = e^{At}v$ solves the initial value problem $x' = Ax$ with $x(0) = v$.
- Refined strategy: Compute $e^{At}v$ for $n$ linearly independent vectors $v$.
  - Computing $e^{At}v$ is hard except for specially chosen vectors $v$.

Key to Computing $e^{At}v$

Suppose that $A$ an $n \times n$ matrix, and $\lambda$ a number (an eigenvalue). Then

$$e^{At}v = e^{\lambda t} \cdot e^{(A-\lambda I)v} = e^{\lambda t} \cdot [v + t(A - \lambda I)v + \frac{t^2}{2!}(A - \lambda I)^2v + \cdots]$$

- If $\lambda$ is an eigenvalue and $v$ is an associated eigenvector, then $(A - \lambda I)v = 0$, so $e^{At}v = e^{\lambda t}v$.
- If $(A - \lambda I)^2v = 0$, then $e^{At}v = e^{\lambda t}[v + t(A - \lambda I)v]$.

Generalized Eigenvectors

Definition: If $\lambda$ is an eigenvalue of $A$ and $(A - \lambda I)^p v = 0$ for some integer $p \geq 1$, then $v$ is called a generalized eigenvector associated with $\lambda$.

- Then
  $$e^{At}v = e^{\lambda t} \left[ v + t(A - \lambda I)v + \frac{t^2}{2!}(A - \lambda I)^2v + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}v \right]$$
- We can compute $e^{At}v$ for any generalized eigenvector.
**Multiplicities**

A an \( n \times n \) matrix with distinct eigenvalues \( \lambda_1, \ldots, \lambda_k \).

- The characteristic polynomial has the form
  \[
p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}.
\]
- The algebraic multiplicity of \( \lambda_j \) is \( q_j \).
- \( q_1 + q_2 + \ldots + q_k = n \).
- The geometric multiplicity of \( \lambda_j \) is \( d_j \), the dimension of the eigenspace of \( \lambda_j \).
- \( 1 \leq d_j \leq q_j \).
- There is an integer \( k_j \leq q_j \) for which \( \text{null}((A - \lambda_j I)^{k_j}) \) has dimension \( q_j \).

**Procedure for Solving \( x' = Ax \)**

- Find the eigenvalues of \( A \) and their algebraic multiplicities.
- For each eigenvalue \( \lambda \) with algebraic multiplicity \( q \):
  - Find the smallest integer \( k \) for which \( \text{null}((A - \lambda I)^k) \) has dimension \( q \).
  - Find a basis for \( \text{null}((A - \lambda I)^k) \).
  - For each vector \( v \) in the basis compute the solution \( x(t) = e^{\lambda t}v \).
- The set of all of these solutions is a fundamental set of solutions.

**Replacing Complex Solutions with Real Solutions**

- If \( A \) has complex eigenvalues, the fundamental set of solutions contains complex valued solutions.
- Complex solutions occur in complex conjugate pairs \( z(t) = x(t) + iy(t) \) and \( \bar{z}(t) = x(t) - iy(t) \).
- Replace \( z(t) \) and \( \bar{z}(t) \) with the real solutions \( x(t) = \text{Re}(z(t)) \) and \( y(t) = \text{Im}(z(t)) \).
Solutions to Higher Order Equations

Homogenous linear equation with constant coefficients:
\[ y'' + py' + qy = 0 \]

- Look for exponential solutions \( y(t) = e^{\lambda t} \).
- Characteristic polynomial: \( \lambda^2 + p\lambda + q \).
- If \( \lambda \) is a root of the characteristic polynomial then \( y(t) = e^{\lambda t} \) is a solution.

Fundamental sets of solutions

- Two distinct real roots \( \lambda_1 \) and \( \lambda_2 \):
  \[ y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}. \]
- One real root \( \lambda \) of multiplicity 2:
  \[ y_1(t) = e^{\lambda t} \quad \text{and} \quad y_2(t) = te^{\lambda t}. \]
- Complex conjugate roots \( \lambda = \alpha \pm i\beta \):
  \[ y_1(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y_2(t) = e^{\alpha t} \sin \beta t. \]

Inhomogeneous Equations

\[ y'' + py' + qy = f(t) \]

- The method of undetermined coefficients finds a particular solution \( y_p(t) \).
- The general solution is
  \[ y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t). \]
  
  where \( y_1 \) and \( y_2 \) are a fundamental set of solutions to the homogeneous equation.
- If the forcing term \( f(t) \) has a form which is replicated under differentiation, look for a particular solution of the same general form as the forcing term.
Cases

- If \( f(t) = Ce^{kt} \), try \( y_p(t) = ae^{kt} \).
- If \( f(t) = A \cos \omega t + B \sin \omega t \), try \( y_p(t) = a \cos \omega t + b \sin \omega t \).
  - Or try the complex method.
- If \( f(t) \) is a polynomial of degree \( n \), let \( y_p \) be a polynomial of degree \( n \).
- Exceptional cases: Multiply expected form of \( y_p \) by \( t \).
- Combination cases: Solve the equation in pieces.

Harmonic Motion

- Spring: \( y'' + \frac{2}{m}y' + \frac{1}{m^2}y = \frac{1}{m}F(t) \).
- Circuit: \( I'' + \frac{2R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t) \).
- Essentially the same equation. Use \( x'' + 2cx' + \omega_0^2 x = f(t) \).
  - We call this the equation for harmonic motion.
- \( \omega_0 \) is the natural frequency. \( c \) is the damping constant. \( f(t) \) is the forcing term.

Unforced Harmonic Motion

\[ x'' + 2cx' + \omega_0^2 x = 0 \]

- Undamped: \( c = 0 \).
- Underdamped: \( 0 < c < \omega_0 \).
- Critically damped: \( c = \omega_0 \).
- Over damped: \( c > \omega_0 \).
Forced Harmonic Motion

\[ x'' + 2cx' + \omega_0^2 x = A \cos \omega t \]

- \( A \) is the forcing amplitude and \( \omega \) is the forcing frequency.
- The general solution is \( x(t) = x_p(t) + x_h(t) \).
  - \( x_p \) is a particular solution. \( x_h \) is the general solution of the homogenous equation.
- Undamped: \( c = 0 \).
  - \( \omega \neq \omega_0 \): Beats.
  - \( \omega = \omega_0 \): Resonance.

Forced, Damped Harmonic Motion

\[ x'' + 2cx' + \omega_0^2 x = A \cos \omega t \]

- \( c > 0 \) implies that \( x_h(t) \to 0 \) as \( t \) increases, so \( x_h \) is called the transient term.
- \( x_p(t) \) is called the steady-state solution. It has the form
  \[ x_p(t) = G(\omega)A \cos(\omega t - \phi) \]
  - \( x_p \) is oscillatory at the driving frequency.
  - The amplitude of \( x_p \) is the product of the gain, \( G(\omega) \), and the amplitude of the forcing function.
  - \( x_p \) has a phase shift of \( \phi \) with respect to the forcing function.

Qualitative Analysis

- Existence and uniqueness.
- For an autonomous system \( x' = f(x) \), the basic question is, What happens to all solutions as \( t \to \infty \)?
- The easy cases: equilibrium points \( f(x_0) = 0 \) and equilibrium solutions \( x(t) = x_0 \).
- Local qualitative analysis: What happens as \( t \to \infty \) to all solutions that start near an equilibrium point \( x_0 \)?
  - This is the question of stability.
- Global qualitative analysis: What happens to all solutions as \( t \to \infty \)?
Stability

Suppose the autonomous system $x' = f(x)$ has an equilibrium point at $x_0$.

- $x_0$ is stable if every solution that starts close to $x_0$ stays close to $x_0$.
- $x_0$ is asymptotically stable if every solution that starts close to $x_0$ stays near $x_0$ and approaches $x_0$ as $t \to \infty$.
- $x_0$ is called a sink.
- $x_0$ is unstable if there are solutions starting arbitrarily close to $x_0$ that move away from $x_0$.

Stability for $x' = Ax$

- $D = 2$: Trace-determinant plane.
- **Theorem:** Let $A$ be an $n \times n$ real matrix.
  - Suppose the real part of every eigenvalue of $A$ is negative. Then $0$ is an asymptotically stable equilibrium point for the system $x' = Ax$.
  - Suppose $A$ has at least one eigenvalue with positive real part. Then $0$ is an unstable equilibrium point for the system $x' = Ax$.

Stability for $x' = f(x)$

- Suppose that $x_0$ is an equilibrium point.
- The **linearization** at $x_0$ is the system $u' = Ju$, where $J$ is the Jacobian matrix of $f$ at $x_0$.
- For the planar system $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$, the Jacobian is
  \[
  J = \begin{pmatrix}
  \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
  \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
  \end{pmatrix}
  \]
Stability for $D = 2$

Theorem: Consider the planar system

\[
x' = f(x, y) \\
y' = g(x, y)
\]

where $f$ and $g$ are continuously differentiable. Suppose that $(x_0, y_0)$ is an equilibrium point. If the linearization at $(x_0, y_0)$ has a generic equilibrium point at the origin, then the equilibrium point at $(x_0, y_0)$ is of the same type.

Stability for $D \geq 1$

Theorem: Suppose that $y_0$ is an equilibrium point for $y' = f(y)$. Let $J$ be the Jacobian of $f$ at $y_0$.

1. Suppose that the real part of every eigenvalue of $J$ is negative. Then $y_0$ is an asymptotically stable equilibrium point.

2. Suppose that $J$ has at least one eigenvalue with positive real part. Then $y_0$ is an unstable equilibrium point.

Global Geometric Analysis

- What happens to all solutions as $t \to \infty$?
- The (forward) limit set of the solution $y(t)$ that starts at $y_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(y_0)$.
  - $x \in \omega(y_0)$ if there is a sequence $t_k \to \infty$ such that $y(t_k) \to x$.
- What is $\omega(y_0)$ for all $y_0$?
  - What is the limit set for all solutions?
- In dimension 1, all limit sets are equilibrium points.
Limit Sets in Dimension 2

**Theorem:** If $S$ is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbb{R}^2$, then $S$ is one of the following:

- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the Poincaré-Bendixson alternatives.
- In dimension 3 the answer is unknown.

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Invariant Sets

**Definition:** A set $S$ is (positively) invariant for the system $y' = f(y)$ if $y(0) = y_0 \in S$ implies that $y(t) \in S$ for all $t \geq 0$.

- Examples include equilibrium points, and any solution curve.
- In dimension 2, invariant sets can frequently be found using:
  - nullclines,
  - polar coordinates.

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Poincaré-Bendixson Theorem

**Theorem:** Suppose that $R$ is a closed and bounded planar region that is positively invariant for a planar system. If $R$ contains no equilibrium points, then there is a closed solution curve in $R$.

- The theorem is also true if the set $R$ is negatively invariant.
- The closed solution curve might be a limit cycle.
Solving Separable Equations

\[ \frac{dy}{dt} = g(y)h(t) \]

The three step solution process:
1. Separate the variables. \( \frac{dy}{g(y)} = h(t) \, dt \) if \( g(y) \neq 0 \).
2. Integrate both sides. \( \int \frac{dy}{g(y)} = \int h(t) \, dt \).
3. Solve for \( y(t) \).

Solving the Linear Equation

\[ x' = a(t)x + f(t) \]

Four step process:
1. Rewrite as \( x' - ax = f \).
2. Multiply by the integrating factor
   \( u(t) = e^{-\int a(t) \, dt} \).
   Equation becomes \( [ux]' = ux' - ax = uf \).
3. Integrate: \( u(t)x(t) = \int u(t)f(t) \, dt + C \).
4. Solve for \( x(t) \).

Eigenvalues and Eigenvectors

- \( \lambda \) is an eigenvalue of \( A \) if there is a nonzero vector \( v \) such that \( Av = \lambda v \). If \( \lambda \) is an eigenvalue of \( A \), then any vector \( v \) such that \( Av = \lambda v \) is called an eigenvector associated with \( \lambda \).
- \( \lambda \) is an eigenvalue of \( A \) \( \iff \) \( \det(A - \lambda I) = 0 \).
  - \( p(\lambda) = \det(A - \lambda I) \) is called the characteristic polynomial of \( A \).
- \( v \) is an eigenvector associated with the eigenvalue \( \lambda \) \( \iff \) \( v \in \text{null}(A - \lambda I) \).
  - \( \text{null}(A - \lambda I) \) is called the eigenspace of \( \lambda \).