Math 211

Review for the Final Exam

December 8, 2002
The Final Exam

- The final will be comprehensive, covering material from the entire semester.
- The final will emphasize the material covered since the last exam.
- These slides will cover primarily the material covered since the last exam. They do *not* cover all of the material on the exam.
- Questions about any of the material of the course will be answered.
The Themes of the Course

• Modeling.
  ♦ Population, finance, mixing, motion, vibrating spring, electrical circuits, . . .

• Exact solutions.
  ♦ Separable and linear equations in dimension 1.
  ♦ Linear equations in higher dimension.
    ▶ Matrix algebra.

• Second order equations.

• Numerical solutions.

• Geometric analysis.
Solving $x' = Ax$

- $A$ is an $n \times n$ matrix.

- Solution strategy: Look for a fundamental set of solutions, i.e., $n$ linearly independent solutions.

- The function $x(t) = e^{tA}v$ solves the initial value problem $x' = Ax$ with $x(0) = v$.

- Refined strategy: Compute $e^{tA}v$ for $n$ linearly independent vectors $v$.

  - Computing $e^{tA}v$ is hard except for specially chosen vectors $v$. 
Suppose that $A$ an $n \times n$ matrix, and $\lambda$ a number (an eigenvalue). Then

$$e^{tA}v = e^{\lambda t} \cdot e^{t(A-\lambda I)}v$$

$$= e^{\lambda t} \cdot [v + t(A - \lambda I)v + \frac{t^2}{2!}(A - \lambda I)^2v + \cdots]$$

- If $\lambda$ is an eigenvalue and $v$ is an associated eigenvector, then $(A - \lambda I)v = 0$, so $e^{tA}v = e^{\lambda t}v$.
- If $(A - \lambda I)^2v = 0$, then $e^{tA}v = e^{\lambda t}[v + t(A - \lambda I)v]$. 
Generalized Eigenvectors

Definition: If \( \lambda \) is an eigenvalue of \( A \) and
\[(A - \lambda I)^p v = 0\]
for some integer \( p \geq 1 \), then \( v \) is called a generalized eigenvector associated with \( \lambda \).

* Then

\[ e^{tA}v = e^{\lambda t} \left[ v + t(A - \lambda I)v + \frac{t^2}{2!} (A - \lambda I)^2 v + \cdots + \frac{t^{p-1}}{(p-1)!} (A - \lambda I)^{p-1} v \right] \]

* We can compute \( e^{tA}v \) for any generalized eigenvector.
\section*{Multiplicities}

A an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.

- The characteristic polynomial has the form
  \[ p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}. \]

- The \textit{algebraic multiplicity} of $\lambda_j$ is $q_j$.
  \[ q_1 + q_2 + \ldots + q_k = n. \]

- The \textit{geometric multiplicity} of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$.
  \[ 1 \leq d_j \leq q_j. \]

- There is an integer $k_j \leq q_j$ for which $\text{null}((A - \lambda_j I)^{k_j})$ has dimension $q_j$. 

Return Generalized eigenvectors Strategy
Procedure for Solving $x' = Ax$

- Find the eigenvalues of $A$ and their algebraic multiplicities.

- For each eigenvalue $\lambda$ with algebraic multiplicity $q$:
  - Find the smallest integer $k$ for which $\text{null}((A - \lambda I)^k)$ has dimension $q$.
  - Find a basis for $\text{null}((A - \lambda I)^k)$.
  - For each vector $v$ in the basis compute the solution $x(t) = e^{tA}v$.

- The set of all of these solutions is a fundamental set of solutions.
Replacing Complex Solutions with Real Solutions

- If \( A \) has complex eigenvalues, the **fundamental** set of solutions contains complex valued solutions.

- Complex solutions occur in complex conjugate pairs
  \[ z(t) = x(t) + iy(t) \text{ and } \overline{z(t)} = x(t) - iy(t). \]

- Replace \( z(t) \) and \( \overline{z(t)} \) with the real solutions
  \[ x(t) = \text{Re}(z(t)) \text{ and } y(t) = \text{Im}(z(t)). \]
Solutions to Higher Order Equations

Homogeneous linear equation with constant coefficients:

\[ y'' + py' + qy = 0 \]

- Look for exponential solutions \( y(t) = e^{\lambda t} \).
- **Characteristic polynomial**: \( \lambda^2 + p\lambda + q \).
- If \( \lambda \) is a root of the characteristic polynomial then \( y(t) = e^{\lambda t} \) is a solution.
Fundamental sets of solutions

- Two distinct real roots $\lambda_1$ and $\lambda_2$:
  \[ y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}. \]

- One real root $\lambda$ of multiplicity 2:
  \[ y_1(t) = e^{\lambda t} \quad \text{and} \quad y_2(t) = te^{\lambda t}. \]

- Complex conjugate roots $\lambda = \alpha \pm i\beta$:
  \[ y_1(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y_2(t) = e^{\alpha t} \sin \beta t. \]
Inhomogeneous Equations

\[ y'' + Py' + Qy = f(t) \]

- The method of undetermined coefficients finds a particular solution \( y_p(t) \).
- The general solution is

\[ y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t), \]

where \( y_1 \) and \( y_2 \) are a fundamental set of solutions to the homogeneous equation.
- If the forcing term \( f(t) \) has a form which is replicated under differentiation, look for a particular solution of the same general form as the forcing term.
Cases

- If \( f(t) = Ce^{bt} \), try \( y_p(t) = ae^{bt} \).
- If \( f(t) = A \cos \omega t + B \sin \omega t \), try \( y_p(t) = a \cos \omega t + b \sin \omega t \).
  - Or try the complex method.
- If \( f(t) \) is a polynomial of degree \( n \), let \( y_p \) be a polynomial of degree \( n \).
- Exceptional cases: Multiply expected form of \( y_p \) by \( t \).
- Combination cases: Solve the equation in pieces.
Harmonic Motion

- Spring: $y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t)$.  
- Circuit: $I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t)$.  
- Essentially the same equation. Use  
  \[ x'' + 2cx' + \omega_0^2x = f(t). \]

- We call this the equation for harmonic motion.

- $\omega_0$ is the natural frequency. $c$ is the damping constant. $f(t)$ is the forcing term.
Unforced Harmonic Motion

\[ x'' + 2cx' + \omega_0^2 x = 0 \]

- **Undamped**: \( c = 0 \).
- **Underdamped**: \( 0 < c < \omega_0 \).
- **Critically damped**: \( c = \omega_0 \).
- **Over damped**: \( c > \omega_0 \).
Forced Harmonic Motion

\[ x'' + 2cx' + \omega_0^2 x = A \cos \omega t \]

- \( A \) is the \textit{forcing amplitude} and \( \omega \) is the \textit{forcing frequency}.

- The \textit{general solution} is \( x(t) = x_p(t) + x_h(t) \).
  - \( x_p \) is a \textit{particular solution}. \( x_h \) is the \textit{general solution} of the homogenous equation.

- Undamped: \( c = 0 \).
  - \( \omega \neq \omega_0 \): Beats.
  - \( \omega = \omega_0 \): Resonance.
Forced, Damped Harmonic Motion

\[ x'' + 2cx' + \omega_0^2 x = A \cos \omega t \]

- \( c > 0 \) implies that \( x_h(t) \to 0 \) as \( t \) increases, so \( x_h \) is called the \textit{transient term}.

- \( x_p(t) \) is called the \textit{steady-state solution}. It has the form

\[ x_p(t) = G(\omega)A \cos(\omega t - \phi) \]

- \( x_p \) is oscillatory at the driving frequency.

- The amplitude of \( x_p \) is the product of the \textit{gain}, \( G(\omega) \), and the amplitude of the forcing function.

- \( x_p \) has a \textit{phase shift} of \( \phi \) with respect to the forcing function.
Qualitative Analysis

• Existence and uniqueness.

• For an autonomous system $x' = f(x)$, the basic question is, What happens to all solutions as $t \to \infty$?

• The easy cases: equilibrium points $f(x_0) = 0$ and equilibrium solutions $x(t) = x_0$.

• Local qualitative analysis: What happens as $t \to \infty$ to all solutions that start near an equilibrium point $x_0$?
  ♦ This is the question of stability.

• Global qualitative analysis: What happens to all solutions as $t \to \infty$?
Stability

Suppose the autonomous system \( x' = f(x) \) has an equilibrium point at \( x_0 \).

- \( x_0 \) is **stable** if every solution that starts close to \( x_0 \) stays close to \( x_0 \).

- \( x_0 \) is **asymptotically stable** if every solution that starts close to \( x_0 \) stays near \( x_0 \) and approaches \( x_0 \) as \( t \to \infty \).
  - \( x_0 \) is called a **sink**.

- \( x_0 \) is **unstable** if there are solutions starting arbitrarily close to \( x_0 \) that move away from \( x_0 \).
Stability for $x' = Ax$

- $D = 2$: Trace-determinant plane.

- **Theorem:** Let $A$ be an $n \times n$ real matrix.
  
  - Suppose the real part of every eigenvalue of $A$ is negative. Then $0$ is an **asymptotically stable** equilibrium point for the system $x' = Ax$.
  
  - Suppose $A$ has at least one eigenvalue with positive real part. Then $0$ is an **unstable** equilibrium point for the system $x' = Ax$. 
Stability for $x' = f(x)$

- Suppose that $x_0$ is an equilibrium point.
- The *linearization* at $x_0$ is the system $u' = Ju$, where $J$ is the *Jacobian matrix* of $f$ at $x_0$.
- For the planar system \[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\] the Jacobian is

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
\]
Stability for $D = 2$

**Theorem:** Consider the planar system

\[
x' = f(x, y)\\
y' = g(x, y)
\]

where $f$ and $g$ are continuously differentiable. Suppose that $(x_0, y_0)$ is an equilibrium point. If the linearization at $(x_0, y_0)$ has a generic equilibrium point at the origin, then the equilibrium point at $(x_0, y_0)$ is of the same type.
Stability for $D \geq 1$

**Theorem:** Suppose that $y_0$ is an equilibrium point for $y' = f(y)$. Let $J$ be the Jacobian of $f$ at $y_0$.

1. Suppose that the real part of every eigenvalue of $J$ is negative. Then $y_0$ is an asymptotically stable equilibrium point.

2. Suppose that $J$ has at least one eigenvalue with positive real part. Then $y_0$ is an unstable equilibrium point.
Global Geometric Analysis

- What happens to all solutions as $t \to \infty$?
- The (forward) limit set of the solution $y(t)$ that starts at $y_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(y_0)$.
  - $x \in \omega(y_0)$ if there is a sequence $t_k \to \infty$ such that $y(t_k) \to x$.
- What is $\omega(y_0)$ for all $y_0$?
  - What is the limit set for all solutions?
- In dimension 1, all limit sets are equilibrium points.
Limit Sets in Dimension 2

**Theorem:** If $S$ is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbb{R}^2$, then $S$ is one of the following:

- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the *Poincaré-Bendixson alternatives*.

- In dimension 3 the answer is unknown.
Invariant Sets

**Definition:** A set $S$ is *(positively) invariant* for the system $y' = f(y)$ if $y(0) = y_0 \in S$ implies that $y(t) \in S$ for all $t \geq 0$.

- Examples include equilibrium points, and any solution curve.
- In dimension 2, invariant sets can frequently be found using:
  - nullclines,
  - polar coordinants.
Poincaré-Bendixson Theorem

**Theorem:** Suppose that $R$ is a closed and bounded planar region that is positively invariant for a planar system. If $R$ contains no equilibrium points, then there is a closed solution curve in $R$.

- The theorem is also true if the set $R$ is negatively invariant.
- The closed solution curve might be a limit cycle.
Solving Separable Equations

\[ \frac{dy}{dt} = g(y)h(t) \]

The three step solution process:

1. Separate the variables. \( \frac{dy}{g(y)} = h(t) \, dt \) if \( g(y) \neq 0 \).

2. Integrate both sides. \( \int \frac{dy}{g(y)} = \int h(t) \, dt \)

3. Solve for \( y(t) \).
Solving the Linear Equation

\[ x' = a(t)x + f(t) \]

Four step process:

1. Rewrite as \( x' - ax = f \).

2. Multiply by the integrating factor

\[
u(t) = e^{-\int a(t) \, dt}.\]

Equation becomes \([ux]' = ux' - aux = uf\).

3. Integrate: \( u(t)x(t) = \int u(t)f(t) \, dt + C \).

4. Solve for \( x(t) \).
Eigenvalues and Eigenvectors

• $\lambda$ is an *eigenvalue* of $A$ if there is a nonzero vector $v$ such that $Av = \lambda v$. If $\lambda$ is an eigenvalue of $A$, then any vector $v$ such that $Av = \lambda v$ is called an *eigenvector associated with* $\lambda$.

• $\lambda$ is an eigenvalue of $A \iff \det(A - \lambda I) = 0$.
  - $p(\lambda) = \det(A - \lambda I)$ is called the *characteristic polynomial* of $A$.

• $v$ is an eigenvector associated with the eigenvalue $\lambda \iff v \in \text{null}(A - \lambda I)$.
  - $\text{null}(A - \lambda I)$ is called the *eigenspace* of $\lambda$. 