Math 211

Lecture #3

Solutions to Differential Equations

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Differential Equation:

An equation involving an unknown function and one or more of its derivatives, in addition to the independent variable.

- Example: \( y' = \frac{dy}{dt} = 2ty \)

- General equation: \( y' = \frac{dy}{dt} = f(t, y) \)

- \( t \) is the independent variable.

- \( y = y(t) \) is the unknown function.

- \( y' = 2ty \) is of order 1.
The general first order equation is

\[ y' = f(t, y). \]

A *solution* is a function \( y(t) \), defined for \( t \) in an interval, which is differentiable at each point and satisfies

\[ y'(t) = f(t, y(t)) \]

for every point \( t \) in the interval.
Example: $y' = 2ty$

Is $y(t) = e^{t^2}$ a solution?

- By substitution $y'(t) = 2ty(t)$, so $y(t) = e^{t^2}$ is a solution.

Is $y(t) = e^t$ a solution?

- By substitution $y'(t) \neq 2ty(t)$, so $y(t) = e^t$ is not a solution to the equation $y' = 2ty$.

Verification by substitution is always available.
More about Solutions

- A solution is a function. What is a function?
  - An exact, algebraic formula (e.g., $y(t) = e^{t^2}$).
  - A convergent power series.
  - The limit of a sequence of functions.
- An ODE is a function generator.
- Two of the themes of the course are aimed at those solutions for which there is no exact formula.
An ODE is a Function Generator

Example: \( y' = y^2 - t, \quad y(0) = 0 \)

- There is no solution to this IVP which can be given using a formula.
- Nevertheless, there is a solution. We can find as many terms in the power series for \( y(t) \) as we want.

\[
y(t) = -\frac{1}{2}t^2 + \frac{1}{20}t^5 - \frac{1}{160}t^8 + \ldots
\]
Particular and General Solutions

For the equation $y' = 2ty$

- $y(t) = \frac{1}{2}e^{t^2}$ is a solution. It is a *particular solution*.
- $y(t) = Ce^{t^2}$ is a solution for any constant $C$. This is a *general solution*.

General solutions contain arbitrary constants. Particular solutions do not.
Initial Value Problem (IVP)

A differential equation & an initial condition.

- Example: Find $y(t)$ with $y' = -2ty$ with $y(0) = 4$.
- General solution: $y(t) = Ce^{-t^2}$.
- Plug in the initial condition:

  $y(0) = 4, \quad Ce^0 = 4, \quad C = 4$

Solution to the IVP: $y(t) = 4e^{-t^2}$. 
Normal Form of an Equation

The first order differential equation

\[ y' = f(t, y) \]

is said to be in *normal form*.

- Example: The differential equation
  \[(1 + t^2)y' + y^2 = t^3\] is not in normal form.

- Solve for \(y'\) to put the equation into normal form:
  \[ y' = \frac{t^3 - y^2}{1 + t^2} \]

- Many statements about differential equations require the equation to be in normal form.
### Interval of Existence

The largest interval over which a solution can exist.

- **Example**: $y' = -2ty$, $y(0) = 4$.
  - The interval of existence is $\mathbb{R} = (-\infty, \infty)$.

- **Example**: $y' = 1 + y^2$ with $y(0) = 1$.
  - General solution: $y(t) = \tan(t + C)$
  - Initial Condition: $y(0) = 1 \Rightarrow y(t) = \tan(t + \pi/4)$
  - The solution exists and is continuous for $-\pi/2 < t + \pi/4 < \pi/2$.
  - The interval of existence is $-3\pi/4 < t < \pi/4$. 
Geometric Interpretation of

\[ y' = f(t, y) \]

If \( y(t) \) is a solution, and \( y(t_0) = y_0 \), then

\[ y'(t_0) = f(t_0, y(t_0)) = f(t_0, y_0). \]

- The slope to the graph of \( y(t) \) at the point \( (t_0, y_0) \) is given by \( f(t_0, y_0) \).

- Imagine a small line segment attached to each point of the \( (t, y) \) plane with the slope \( f(t, y) \).
The Direction Field

\[ x' = x^2 - t \]
Autonomous Equations

- General equation: \( \frac{dy}{dt} = f(t, y) \)
- Autonomous equation: \( \frac{dy}{dt} = f(y) \)
- Examples:
  - \( \frac{dy}{dt} = t - y^2 \) is not autonomous.
  - \( \frac{dy}{dt} = y(1 - y) \) is autonomous.

In an *autonomous equation* the right-hand side has no explicit dependence on the independent variable.
Equilibrium Points

• An *equilibrium point* for the *autonomous equation* \( \frac{dy}{dt} = f(y) \) is a point \( y_0 \) such that \( f(y_0) = 0 \).

• Corresponding to the equilibrium point \( y_0 \) there is the constant *equilibrium solution* \( y(t) = y_0 \).

• Example: \( \frac{dy}{dt} = y(2 - y)/3 \) is an autonomous equation.
  
  ♦ The equilibrium points are \( y_0 = 0 \) or 2.
  
  ♦ The corresponding equilibrium solutions are \( y(t) = 0 \) and \( y(t) = 2 \).
Between Equilibrium Points

- \( \frac{dy}{dt} = f(y) > 0 \Rightarrow y(t) \) is increasing.
- \( \frac{dy}{dt} = f(y) < 0 \Rightarrow y(t) \) is decreasing.
- The graphs of solutions to first order equations cannot cross (uniqueness theorem).
- Example: \( \frac{dy}{dt} = y(2 - y)/3 \)