Qualitative Analysis

- Do solutions always exist?
- Do solutions to an initial value problem always exist?
- How many solutions are there?
- How many solutions are there to an initial value problem?
- Can we predict the behavior of solutions without having a formula?

Existence Theorem

**Theorem:** Suppose the function \( f(t, y) \) is defined and continuous in the rectangle \( R \) in the \( ty \)-plane. Then given any point \((t_0, y_0) \in R\), the initial value problem

\[
g'(t) = f(t, g(t)) \quad \text{with} \quad g(t_0) = y_0
\]

has a solution \( g(t) \) defined in an interval containing \( t_0 \).

Furthermore, the solution will be defined at least until the solution curve \( t \mapsto (t, g(t)) \) leaves the rectangle \( R \).
Existence Theorem

- **Hypotheses:**
  - The equation is in normal form $y' = f(t, y)$.
  - The right hand side, $f(t, y)$, is continuous in the rectangle $R$.
  - The initial point $(t_0, y_0)$ is in the rectangle $R$.

- **Conclusions:**
  - There is a solution starting at the initial point.
  - The solution is defined at least until the solution curve $t \to (t, y(t))$ leaves the rectangle $R$.

Existence of a Solution

- The existence theorem does not guarantee an explicitly defined solution.
- In the proof, the solution is constructed as the limit of a sequence of explicitly defined functions.
- Frequently no explicit formula is possible.
- An ordinary differential equation is a function generator.

Interval of Existence

- Example: $y' = 1 + y^2$ with $y(0) = 0$.
- RHS $f(t, y) = 1 + y^2$ is defined and continuous on the whole $ty$-plane. The rectangle $R$ can be any rectangle in the plane.
- Solution $y(t) = \tan t$ “blows up” at $t = \pm\pi/2$.
- Thus the size of the rectangle on which $f(t, y)$ is continuous does not say much about the interval of existence.
Uniqueness of Solutions

- How many solutions does an initial value problem have?
- The uniqueness of solutions to an initial value problem is the mathematical equivalent of being able to predict results in science and engineering.
- The uniqueness of solutions to a differential equation model is equivalent to a system being causal.

Example of Non-uniqueness

- Initial value problem \( y' = y^{1/3} \) with \( y(0) = 0 \).
- The constant function \( y_1(t) = 0 \) is a solution.
- Solve by separation of variables to find that
  \[
  y_2(t) = \begin{cases}
  \left( \frac{2t}{3} \right)^{3/2}, & \text{if } t > 0 \\
  0, & \text{if } t \leq 0
  \end{cases}
  \]
  is also a solution.

Uniqueness Theorem

**Theorem:** Suppose the function \( f(t,y) \) and its partial derivative \( \partial f/\partial y \) are continuous in the rectangle \( R \) in the \( ty \)-plane. Suppose that \( (t_0,y_0) \in R \). Suppose that

\[
  x' = f(t,x) \quad \text{and} \quad y' = f(t,y),
\]

and that

\[
  x(t_0) = y(t_0) = x_0.
\]

Then as long as \( (t,x(t)) \) and \( (t,y(t)) \) stay in \( R \) we have

\[
  x(t) = y(t).
\]
Uniqueness Theorem

• Hypotheses:
  • The equation is in normal form $y' = f(t, y)$.
  • The right hand side, $f(t, y)$, and its derivative $\partial f/\partial y$ are continuous in the rectangle $R$.
  • The initial point $(t_0, y_0)$ is in the rectangle $R$.

• Conclusions:
  • There is one and only one solution starting at the initial point.
  • The solution is defined at least until the solution curve $t \to (t, y(t))$ leaves the rectangle $R$.

Geometric Interpretation

• Solution curves cannot cross.
• They cannot even touch at one point.
• $y' = (y - 1)(\cos t - y)$ and $y(0) = 2$. Show that $y(t) > 1$ for all $t$.
• $y' = y - (1 - t)^2$ and $y(0) = 0$. Show that $y(t) < 1 + t^2$ for all $t$.

E & U for Linear Equations

Theorem: Suppose that $a(t)$ and $g(t)$ are continuous on an interval $I = (a, b)$. Then given $t_0 \in I$ and any $y_0$, the initial value problem

$$y' = a(t)y + g(t) \quad \text{with} \quad y(t_0) = y_0$$

has a unique solution $y(t)$ which exists for all $t \in I$.

• Notice that the RHS is

$$f(t, y) = a(t)y + g(t), \quad \text{and} \quad \partial f/\partial y = a(t).$$

These are continuous for $t \in I$ and all $y$. 
Get a geometric look at existence and uniqueness.

Theorem: Suppose $f(t, y), \frac{\partial f}{\partial y}$ are continuous in the rectangle $R$. Let

$$M = \max_{(t, y) \in R} \left| \frac{\partial f}{\partial y}(t, y) \right|.$$ 

Suppose that $(t_0, x_0)$ and $(t_0, y_0)$ both lie in $R$, and

$$x' = f(t, x), \quad x(t_0) = x_0 \quad \text{&} \quad y' = f(t, y), \quad y(t_0) = y_0.$$ 

Then as long as $(t, x(t))$ and $(t, y(t))$ stay in $R$ we have

$$|x(t) - y(t)| \leq |x_0 - y_0|e^{M|t-t_0|}.$$ 

Continuity in Initial Conditions

- Inequality: $|x(t) - y(t)| \leq |x_0 - y_0|e^{M|t-t_0|}$.
- The good news:
  - By making sure that $x_0$ and $y_0$ are very close we can make the solutions $x(t)$ and $y(t)$ close for $t$ in an interval containing $t_0$.
  - Solutions are continuous in the initial conditions.
Sensitivity with Respect to Initial Conditions

- Inequality: \[ |x(t) - y(t)| \leq |x_0 - y_0| e^{M|t - t_0|}. \]

- The bad news:
  - As \(|t - t_0|\) increases the RHS grows exponentially.
  - Over long intervals in \(t\) the solutions can get very far apart. Solutions are sensitive to initial conditions.