Math 211

Lecture #9

Qualitative Analysis

September 16, 2002

Uniqueness Theorem

Theorem: Suppose the function \( f(t, y) \) and its partial derivative \( \frac{\partial f}{\partial y} \) are continuous in the rectangle \( R \) in the \( ty \)-plane. Suppose that \( (t_0, x_0) \in R \). Suppose that
\[
x' = f(t, x) \quad \text{and} \quad y' = f(t, y),
\]
and that
\[
x(t_0) = y(t_0) = x_0.
\]
Then as long as \( (t, x(t)) \) and \( (t, y(t)) \) stay in \( R \) we have
\[
x(t) = y(t).
\]

Theorem: Suppose \( f(t, y) \), \( \frac{\partial f}{\partial y} \) are continuous in the rectangle \( R \). Let
\[
M = \max_{(t,y) \in R} \left| \frac{\partial f}{\partial y}(t,y) \right|.
\]
Suppose that \( (t_0, x_0) \) and \( (t_0, y_0) \) both lie in \( R \), and
\[
x' = f(t, x), \quad x(t_0) = x_0 \quad \text{and} \quad y' = f(t, y), \quad y(t_0) = y_0.
\]
Then as long as \( (t, x(t)) \) and \( (t, y(t)) \) stay in \( R \) we have
\[
|x(t) - y(t)| \leq |x_0 - y_0|e^{M|t-t_0|}.
\]
Continuity in Initial Conditions

- Inequality: \(|x(t) - y(t)| \leq |x_0 - y_0|e^{Mt - t_0} - e^{Mt - t_0}.

- The good news:
  - By making sure that \(x_0\) and \(y_0\) are very close we can make the solutions \(x(t)\) and \(y(t)\) close for \(t\) in an interval containing \(t_0\).
  - Solutions are continuous in the initial conditions.

Sensitivity with Respect to Initial Conditions

- Inequality: \(|x(t) - y(t)| \leq |x_0 - y_0|e^{Mt - t_0} - e^{Mt - t_0}.

- The bad news:
  - As \(|t - t_0|\) increases the RHS grows exponentially.
  - Over long intervals in \(t\) the solutions can get very far apart. Solutions are sensitive to initial conditions.

Qualitative Analysis of Autonomous Equations

- Ways to discover the properties of solutions without solving the equation.
  - What happens to solutions \(y(t)\) as \(t \to \infty\).
  - Properties of autonomous equations, \(y' = f(y)\).
    - The direction field does not depend on \(t\).
    - Solution curves can be translated left and right to get other solution curves. i.e., if \(y(t)\) a solution so is \(y_1(t) = y(t + c)\) for any constant \(c\).
Equilibrium Points & Solutions

Autonomous equation: \( y' = f(y) \).

- Equilibrium point: \( f(y_0) = 0 \).
- Equilibrium solution: \( y(t) = y_0 \).

Example: \( y' = \sin y \)

\( \sin y = 0 \iff y = k\pi, \ k = 0, \pm 1, \ldots \)

\( y' = \sin y \) has infinitely many equilibrium solutions:

\( y_k(t) = k\pi \) for \( k = 0, \pm 1, \pm 2, \ldots \)

Between the Equilibrium Points

Example: \( y' = \sin y \).

\( 0 < y < \pi \)

\( y'(t) = \sin y(t) > 0 \Rightarrow y(t) \) is increasing

- By uniqueness, \( 0 < y(t) < \pi \) for all \( t \).
- \( \Rightarrow y(t) \searrow \pi \) as \( t \to \infty \) and \( y(t) \nearrow 0 \) as \( t \to -\infty \)

\( -\pi < y < 0 \)

\( y'(t) = \sin y(t) < 0 \Rightarrow y(t) \) is decreasing

- By uniqueness, \( 0 > y(t) > -\pi \) for all \( t \).
- \( \Rightarrow y(t) \searrow -\pi \) as \( t \to \infty \) and \( y(t) \nearrow 0 \) as \( t \to -\infty \)

Stable & Unstable EPs

An equilibrium point \( y_0 \) is

- *asymptotically stable* if all solutions starting near \( y_0 \) converge to \( y_0 \) as \( t \to \infty \).
- *unstable* if there are solutions starting arbitrarily close to \( y_0 \) which move away from \( y_0 \) as \( t \) increases.

There are 4 possibilities:
A Phase Line for $y' = f(y)$

- A phase line is a $y$-axis, showing
  - the equilibrium points and
  - the direction of the flow between the equilibrium points.
- Examples:
  - The $y$-axis in the plot of $y \rightarrow f(y)$.
  - The $y$-axis in the $ty$-plane where solutions are plotted.

Example – Terminal Velocity

- Assume the magnitude of the resistance is proportional to the square of the velocity:
  \[ v' = -g - k|v|v/m \]
- One equilibrium point at
  \[ v_{\text{term}} = -\sqrt{mg/k}. \]
- $v_{\text{term}}$ is asymptotically stable.

Qualitative Analysis of $y' = f(y)$.

1. Graph $y \rightarrow f(y)$. 

![Graph of y → f(y)](image)
Qualitative Analysis of $y' = f(y)$.

2. Find the equilibrium points where $f(y) = 0$.

3. Determine the behavior between eq. pts.

4. Analyze the equilibrium points.
Qualitative Analysis of $y' = f(y)$.

5. Transfer the phase line to $ty$-space.

Qualitative Analysis of $y' = f(y)$.

6. Plot the equilibrium solutions.

Qualitative Analysis of $y' = f(y)$.

7. Plot other solutions approximately.
Seven Steps

1. Graph $y \rightarrow f(y)$.
2. Find the equilibrium points where $f(y) = 0$.
3. Determine the behavior between eq. pts.
4. Analyze the equilibrium points.
5. Transfer the phase line to $fy$-space.
6. Plot the equilibrium solutions.
7. Plot other solutions approximately.