Math 211

Lecture #10

Population Models

September 18, 2002
Modeling Population

• Assume population changes due to births and deaths only.

• Births are roughly proportional to population, \( B = bP \)
  
  ♦ \( b \) is the \textit{birth rate}. It is the average number of births per individual in one unit of time.

• Deaths are roughly proportional to population, \( D = dP \).
  
  ♦ \( d \) is the \textit{death rate}. It is the probability that any one individual will die in one unit of time.
Modeling Population (cont.)

- Rate of change = births − deaths

\[
\frac{dP}{dt} = B - D = bP - dP = rP
\]

- \( r = b - d \) is the \textit{reproductive rate}.

- In general, \( b \) and \( d \), and therefore \( r \), are not constants.
  - They can depend on \( P \), and perhaps also on \( t \).
The Malthusian Model

• If there exist sufficient resources in term of nutrients and space, \( b \) and \( d \) will be almost constant. Then the reproductive rate \( r = b - d \) is almost a constant.

• If \( r \) is constant we have the **Malthusian model**.

\[
\frac{dP}{dt} = rP \quad \text{with} \quad P(0) = P_0
\]

• Solution: \( P(t) = P_0 e^{rt} \)
  
  ♦ If \( r = b - d > 0 \), \( P(t) \) grows exponentially.
  
  ♦ If \( r = b - d < 0 \), \( P(t) \) decays exponentially.
The Malthusian Model (cont.)

Under what circumstances could the Malthusian model be a good model?

- Requires unlimited resources.
  - OK in laboratory experiments with small populations.
- Populations always outgrow the Malthusian model.
  This was the point that was made by Malthus.
The Logistic Model

- As the population increases individuals compete for resources — for food and for space.

- The birth rate $b$ is the average number of births per individual in one unit of time.

- As $P \uparrow$, $b \downarrow$ because of competition.
  - Competition results from encounters.
  - The number of encounters by one individual is roughly proportional to $P$.
  - $\Rightarrow$ decrease in the birth rate is $\sim P$
  - Assume that $b = b_0 - b_1 P$
The Logistic Model (cont.)

- Increase in the death rate $d$ is $\sim P$
  - Assume that $d = d_0 + d_1 P$
- The reproductive rate is
  
  $$r = b - d = (b_0 - b_1 P) - (d_0 + d_1 P) = r_0 - r_1 P$$

- The result is the logistic model
  
  $$\frac{dP}{dt} = rP = (r_0 - r_1 P)P$$
  
  $$= r_0 \left(1 - \frac{P}{K}\right)P \quad (K = r_0/r_1)$$
Analysis of the Logistic Model

\[ \frac{dP}{dt} = r_0 \left( 1 - \frac{P}{K} \right) P \]

- Equation is autonomous.
- Equilibrium points are 0 & \( K \).
- 0 is unstable, \( K \) is asymptotically stable.
- Any positive solution \( P(t) \to K \) as \( t \to \infty \).
  - \( K \) is the \textit{carrying capacity}.
  - \( r_0 \) is the \textit{reproductive rate at small populations}.
Solution of The Logistic Model

\[
\frac{dP}{dt} = r \left(1 - \frac{P}{K}\right)P \quad \text{with} \quad P(0) = P_0
\]

- Solution:

\[
P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}}
\]
Estimating Parameters

• Malthusian model \( P' = rP \)

\[
P(t) = P_0 e^{rt}
\]

♦ Two parameters \( P_0 \) and \( r \).
♦ Two measurements or observations needed to find the values of \( P_0 \) and \( r \).
♦ It is better to use all of the data and use least squares (linear regression).
Estimating Parameters

- Logistic model: 
  \[ P'(t) = r(1 - P/K)P \]

  \[ P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-rt}} \]

  - Three parameters, \( P_0, r, \) and \( K \).
  - Three measurements or observations needed to find the values of \( P_0, r, \) and \( K \).
  - It is better to use all of the data and use least squares. (Nonlinear regression)
Modelling

- Two ways to write the rate of change of something, e.g., of a population $P$
  - The mathematical way is the derivative, $\frac{dP}{dt}$.
  - The other way involves scientific analysis.,

\[
\frac{dP}{dt} = r \left( 1 - \frac{P}{K} \right) P.
\]

- Setting the two equal gives a differential equation model, in this case the logistic model

\[
\frac{dP}{dt} = r \left( 1 - \frac{P}{K} \right) P
\]
Efficacy of the Logistic Model

• Does a very good job of modeling the growth of populations under controlled circumstances.
  ♦ In laboratory experiments.
  ♦ In other circumstances when the situation does not change.

• For human populations the model always breaks down.
  ♦ Other factors become important, such as immigration, the advance of technology, and changing habits of life.