Compound Interest

Put some money into an account that returns a percentage each year, compounded continuously. How will it grow?

- $P(t)$ is the principal balance in the account, measured in $1000$.
- "Some money" is $P(0) = P_0$.
- "Returns a percentage" is $r\%$/year.
- "Some time later" is measured in years.
- "Compounded continuously" $\Rightarrow P' = rP$.

Compound Interest (cont.)

- Solution
  
  \[ P(t) = P_0 e^{rt} \]

- The principal grows exponentially.
- If $r = 8\%$, then after 20 years
  \[ P(20) = P_0 e^{0.08 \times 20} \]
  \[ = 4.953 P_0 \]
- After 40 years $P(40) = 24.5325 P_0$. 

Math 211
Lecture #11
Financial Models

September 20, 2002
Returns on Investments

What rates of return can we expect?
- Checking accounts — 0 – 3%.
- Money market accounts — 1/4 – 3%.
- Certificates of deposit (3 years) 3 – 4%.
- Industrial bonds — 5.3% (average from 1926 – 2001).
- Stocks — 10.7% (average from 1926 – 2001).

Retirement Account

- Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?
- “A fixed amount each year” is $D$, measured in $1,000 each year. We assume this is invested continuously.

Retirement Account (cont.)

- The model is 
  \[ P' = rP + D. \]
- Solution 
  \[ P(t) = P_0 e^{rt} + \frac{D}{r} [e^{rt} - 1]. \]
Example of a Retirement Account

- Suppose you start with an investment of $1,000 at the age of 25, and invest $100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?

  - $P_0 = 1000$, $D = 100 \times 12 = 1200$, $r = 8\% = 0.08$.

  - At 65 the principal is $377,521$.

  - Is this enough to retire on?

Retirement Planning

- If you need a certain income after you retire, how much must you have in your retirement account when you retire?

  - “Certain income” is $I$ (in $1000/year) withdrawn from the account.

  - “How much” is the amount $P_0$ in the account at retirement.

  - The account still grows due to its return at $r\%$/year.

Retirement Planning (cont.)

- The model is

  $$P' = rP - I, \quad P(0) = P_0.$$ 

- Solution $P(t) = P_0e^{rt} - \frac{I}{r}[e^{rt} - 1]$.

- We are given $I$, $r$, & $P(t_d)$.

- We need to compute $P_0$.  

Retirement Planning – Example 1

• If you will need an income of $75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be $1,043,000.

• How are you going to save over a million dollars?

Retirement Planning (cont.)

• Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance? Include the effect of inflation.

• You starting salary is $S_0$.

• Assume it will increase at $s\%$ per year.

• Then $S' = sS$, or $S(t) = S_0 e^{st}$.

Retirement Planning (cont.)

• The model for the growth of the retirement account is $P' = rP + \lambda S_0 e^{xt}$ with $P(0) = P_0$.

• Solution

$$P(t) = P_0 e^{rt} + \frac{\lambda S_0}{r-s} \left[ e^{rt} - e^{st} \right].$$
Retirement Planning – Example 2

- Assume
  - $P_0 = 1,000$ and $r = 8\%$
  - $S_0 = 35,000$ and $s = 4\%$
  - Notice that $S(40) = 173,356$.
  - Need a retirement income of $150,000$.
  - Aim for a balance at retirement of $2,000,000$.
  - Requires $\lambda = 11.53\%$.

Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at $\lambda\%$, and decays linearly to 0 over 40 years.

\[ P' = rP + \lambda(1 - t/40)S_0e^{st} \]

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grows linearly over 40 years to $\lambda\%$.

\[ P' = rP + \frac{\lambda t}{40}S_0e^{st} \]