Math 211

Lecture #11

Financial Models

September 20, 2002
Compound Interest

Put some money into an account that returns a percentage each year, compounded continuously. How will it grow?

- $P(t)$ is the principal balance in the account, measured in $1000$.
- “Some money” is $P(0) = P_0$.
- “Returns a percentage” is $r\%$/year.
- “Some time later” is measured in years.
- “Compounded continuously” $\Rightarrow P' = rP$. 
Compound Interest (cont.)

- **Solution**

\[ P(t) = P_0 e^{rt} \]

- The principal grows exponentially.

- If \( r = 8\% \), then after 20 years

\[ P(20) = P_0 e^{0.08 \times 20} \]

\[ = 4.953 \ P_0 \]

- After 40 years \( P(40) = 24.5325 \ P_0 \).
What rates of return can we expect?

- Checking accounts — 0 – 3%.
- Money market accounts — 1/4 – 3%.
- Certificates of deposit (3 years) 3 – 4 %.
- Industrial bonds — 5.3% (average from 1926 – 2001).
- Stocks — 10.7% (average from 1926 – 2001).
Retirement Account

- Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?

  - “A fixed amount each year” is $D$, measured in $1,000 each year. We assume this is invested continuously.
• The model is

\[ P' = rP + D. \]

• Solution

\[ P(t) = P_0 e^{rt} + \frac{D}{r} [e^{rt} - 1]. \]
Example of a Retirement Account

- Suppose you start with an investment of $1,000 at the age of 25, and invest $100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?

  \[ P_0 = 1000, \ D = 100 \times 12 = 1200, \ r = 8\% = 0.08. \]

- At 65 the principal is $377,521.

- Is this enough to retire on?
Retirement Planning

• If you need a certain income after you retire, how much must you have in your retirement account when you retire?
  ♦ “Certain income” is $I$ (in $1000/year) withdrawn from the account.
  ♦ “How much” is the amount $P_0$ in the account at retirement.
  ♦ The account still grows due to its return at $r\%$/year.
Retirement Planning (cont.)

- The model is
  \[ P' = rP - I, \quad P(0) = P_0. \]

- Solution \( P(t) = P_0 e^{rt} - \frac{I}{r} [e^{rt} - 1]. \)

- We are given \( I, r, \) & \( P(t_d). \)

- We need to compute \( P_0. \)
Retirement Planning – Example 1

• If you will need an income of $75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be

$1,043,000.

• How are you going to save over a million dollars?
Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance? Include the effect of inflation.

- You starting salary is $S_0$.
- Assume it will increase at $s\%$ per year.
  - Then $S' = sS$, or $S(t) = S_0 e^{st}$. 

The model for the growth of the retirement account is

\[ P' = rP + \lambda S_0 e^{st} \quad \text{with} \quad P(0) = P_0. \]

Solution

\[ P(t) = P_0 e^{rt} + \frac{\lambda S_0}{r - s} \left[ e^{rt} - e^{st} \right]. \]
Retirement Planning – Example 2

- Assume
  - \( P_0 = \$1,000 \) and \( r = 8\% \)
  - \( S_0 = \$35,000 \) and \( s = 4\% \)
    - Notice that \( S(40) = \$173,356 \).
  - Need a retirement income of \$150,000.
  - Aim for a balance at retirement of \$2,000,000.
- Requires \( \lambda = 11.53\% \).
Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at $\lambda\%$, and decays linearly to 0 over 40 years.

\[ P' = rP + \lambda(1 - t/40)S_0e^{st} \]

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grows linearly over 40 years to $\lambda\%$.

\[ P' = rP + \frac{\lambda t}{40}S_0e^{st} \]