Math 211

Lecture #15
Systems of Linear Equations

September 30, 2002

Example

Solve

\begin{align*}
3x - 4y + 5z &= 3 \\
-x + 2y - 2z &= -2
\end{align*}

• Find all solutions.
• Find a systematic method which works for all systems, no matter how large.

Vectors and Matrices

• Introduce the vectors

\[
x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 3 \\ -2 \end{pmatrix}
\]

and the matrix

\[
C = \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix}
\]

• \(x\) is the vector of unknowns, \(b\) is the RHS, and \(C\) is the coefficient matrix.
• We will define the product \(Cx\) so that the system can be written as \(Cx = b\).
Vectors

• A vector is a list of numbers
• 2-vectors, 3-vectors, n-vectors
• Row vectors and column vectors.
• A vector has length and direction
  • Parallel vectors are equal
  • Transpose of a vector, \( v^T \).

Algebra of Vectors

• Addition of Vectors
  • Algebraic view of addition
  • Geometric view of addition
  • Addition of more than two vectors
• Multiplication by a Scalar
  • Algebraic view
  • Geometric view

Linear Combinations of Vectors

• Vectors \( x = (2, -3)^T \) and \( y = (-1, 2)^T \).
• Any vector of the form \( ax + by \) is a linear combination of \( x \) and \( y \).
• \( 3x + 2y = (4, -5)^T \).
• Any 2-vector is a linear combination of \( x \) and \( y \).
• Linear combinations of more than two vectors.
Matrices

- A matrix is a rectangular array of numbers.
- Example

\[
A = \begin{pmatrix}
-1 & 0 & 2 & 6 \\
0 & 3 & -4 & 10 \\
3 & 3 & 2 & -5
\end{pmatrix}
\]

- Size of $A = (3,4)$; 3 rows & 4 columns.
  - 3 row vectors and 4 column vectors.

Linear Combinations and Systems

- The example system can be written as a vector equation

\[
\begin{pmatrix}
3x - 4y + 5z \\
-x + 2y - 2z
\end{pmatrix}
= \begin{pmatrix}
3 \\
-2
\end{pmatrix}
\]

- or

\[
x \begin{pmatrix}
3 \\
-1
\end{pmatrix} +
y \begin{pmatrix}
-4 \\
2
\end{pmatrix} +
z \begin{pmatrix}
5 \\
-2
\end{pmatrix}
= \begin{pmatrix}
3 \\
-2
\end{pmatrix}
\]

- These vectors are the column vectors in the coefficient matrix.

Coefficient Matrix

- The coefficient matrix is

\[
C = \begin{pmatrix}
3 & -4 & 5 \\
-1 & 2 & -2
\end{pmatrix}
\]

- Solving the system of equations $\iff$ finding a linear combination of the columns of the coefficient matrix which is equal to the RHS.


Product of a Matrix with a Vector

- The product of a matrix $A$ and a vector $x$ is the linear combination of the columns of $A$ with the elements of $x$ as coefficients.
- Example:

\[
\begin{pmatrix}
3 & -4 & 5 \\
-1 & 2 & -2
\end{pmatrix}
\begin{pmatrix}
x \\ y \\ z
\end{pmatrix}
= x\begin{pmatrix}
3 \\
-1
\end{pmatrix} + y\begin{pmatrix}
-4 \\
2
\end{pmatrix} + z\begin{pmatrix}
5 \\
-2
\end{pmatrix}
\]

Example

- Thus the system of equations becomes

\[
\begin{pmatrix}
3 & -4 & 5 \\
-1 & 2 & -2
\end{pmatrix}
\begin{pmatrix}
x \\ y \\ z
\end{pmatrix}
= \begin{pmatrix}
3 \\
-2
\end{pmatrix}
\]

or

\[C'x = b\]

Computing the Product of a Matrix and a Vector.

- From the definition.
- A faster way.
  - $A = (a_{ij})$, a $p \times q$ matrix, and $x$, a column $q$-vector.

\[
Ax = y \iff
y_i = \sum_{j=1}^q a_{ij}x_j \quad \text{for } 1 \leq i \leq p.
\]

- $Ax$ is only defined if $A$ has the same number of columns as $x$ has rows.
Algebraic Properties of the Matrix-Vector Product

Suppose $A$ is a matrix, $x$ and $y$ are vectors, and $a$ and $b$ are numbers.

- $A(ax) = a(Ax)$
- $A(x + y) = Ax + Ay$
- $A(ax + by) = aAx + bAy$
- Multiplication by a matrix is a linear operation.

Product of Two Matrices

Suppose $A$ is $n \times p$ and $B$ is $p \times q$.

Write $B$ in terms of its column vectors

$$B = [b_1 \ b_2 \ldots \ b_q]$$

Define the product $AB$ by

$$AB = [Ab_1 \ Ab_2 \ldots \ Ab_q]$$

Algebraic Properties of the Product

Suppose that $A$, $B$, and $C$ are matrices

- $A(BC) = (AB)C$
- $A(B + C) = AB + AC$
- $(B + C)A = BA + CA$
- However $AB \neq BA$ in general
The Identity Matrix

- In dimension 3
  \[ I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]

- \( Ix = x \) for every 3-vector \( x \).
- \( IA = A \) for every matrix \( A \) with 3 rows.
- \( AI = A \) for every matrix \( A \) with 3 columns.