Math 211

Lecture #17

Solving Systems of Equations

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Solving Systems of Equations

- We want to find a way to find the solution set of any system.
- We will build towards the method by looking at a series of examples.
- We will start by solving a $2 \times 2$ system using three different, but closely related methods.
Example 1

\[ x + y = 3 \]
\[ 2x - 3y = 1 \]

- **Method 1:** Solve the first equation for \( x \) and substitute into the second equation.

- **Method 2:** Add \(-2\) times the first equation to the second equation to eliminate \( x \).

- **Method 3:** Use the augmented matrix and add \(-2\) times the first row to the second row.
Comparison

• Using any of these methods we get the simpler system

\[ x + y = 3 \]
\[ -5y = -5 \]

• The simple system has the same solutions as the original system.

• The simple system is very easy to solve
  ♦ Solve the last equation first, \( y = 1 \).
  ♦ Then the first equation, \( x = 2 \).
  ♦ This is called\textit{ backsolving}.
Method of Solution

The method is called *elimination and backsolving*, or *Gaussian elimination*. There are four steps:

1. Write down the augmented matrix.
2. **Eliminate** as many coefficients as possible.
   - This is not well defined yet.
3. Write down the **simplified system**.
4. Solve the simplified system by backsolving.
Example 2

\[ y - 2z = -1 \]
\[ 2x - 3y + 4z = 5 \]
\[ -2z + 5y - 8z = -7 \]

1. Write down the augmented matrix.

\[
\begin{pmatrix}
0 & 1 & -2 & -1 \\
2 & -3 & 4 & 5 \\
-2 & 5 & -8 & -7
\end{pmatrix}
\]
Example 2 — Elimination

2. Eliminate as many coefficients as possible.

♦ Interchange rows 1 and 2. \( R_1 \leftrightarrow R_2 \)

♦ \( R_3 \rightarrow R_3 + R_1 \).

♦ \( R_3 \rightarrow R_3 + (-2) \cdot R_2 \). The result is

\[
\begin{pmatrix}
2 & -3 & 4 & 5 \\
0 & 1 & -2 & -1 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]
Example 2 — Backsolving

3. Write down the simplified system.

\[
2x - 3y + 4z = 5 \\
y - 2z = -1
\]

4. Solve the simplified system by backsolving.

♦ \( z \) is a free variable. Set \( z = t \).

♦ \( y = -1 + 2t \).

♦ \( x = 1 + t \).
Example 2 — Solution Set

The solutions are

\[
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
= \begin{pmatrix}
1 \\
-1 \\
0
\end{pmatrix} + t \begin{pmatrix}
1 \\
2 \\
1
\end{pmatrix}.
\]

- This is a line in \( \mathbb{R}^3 \).
  - \( \mathbb{R}^n \) is the set of all \( n \)-vectors.
Elimination — Equations

We only use operations on the equations which will lead to systems of equations with the same solutions. These are:

- **Add** a multiple of one equation to another.
- **Interchange** two equations.
- **Multiply** an equation by a non-zero number.
Elimination — Row operations

The corresponding operations on the rows of the augmented matrix are called row operations.

- Add a multiple of one row to another.
- Interchange two rows.
- Multiply a row by a non-zero number.
The Goal of Elimination

- How simple can we make the augmented matrix?

\[
\begin{pmatrix}
P & * & * & * & * & * & * & * & * & * & * \\
0 & P & * & * & * & * & * & * & * & * & * \\
0 & 0 & 0 & P & * & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & P & * & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & P & * & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & P & * & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & P & * \\
\end{pmatrix}
\]

- \( P \) is a nonzero number, \( * \) is any number.
Row Echelon Form

- The *pivot* of a row in a matrix is the first non-zero element from the left.

- A matrix is in *row echelon form* if every pivot lies strictly to the right of those in rows above.

- Any matrix can be reduced to row echelon form using the first two of the row operations.

- When an augmented matrix has been reduced to row echelon form, the corresponding system can be easily solved by backsolving.
Reduced Row Echelon Form

- Row echelon form, plus all pivots = 1 and all other entries in a pivot column are 0.

\[
\begin{bmatrix}
1 & 0 & * & 0 & 0 & * & 0 & 0 & * \\
0 & 1 & * & 0 & 0 & * & 0 & 0 & * \\
0 & 0 & 0 & 1 & 0 & * & 0 & 0 & * \\
0 & 0 & 0 & 0 & 1 & * & 0 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & * \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Example 3

\[3x_2 - 4x_3 = -7\]
\[-x_1 + 2x_2 = -3\]
\[3x_1 + 2x_2 + x_3 = 2\]

1. The augmented matrix is

\[
\begin{pmatrix}
0 & 3 & -4 & -7 \\
-1 & 2 & 0 & -3 \\
3 & 2 & 1 & 2
\end{pmatrix}
\]
Example 3 — Elimination

2. Elimination:

- $R_1 \leftrightarrow R_2$.
- $R_3 \rightarrow R_3 + 3 \cdot R_1$.
- $R_3 \rightarrow R_3 + (-\frac{8}{3}) \cdot R_2$.
- $R_3 \rightarrow \frac{3}{35} \cdot R_3$.

- We get

\[
\begin{pmatrix}
-1 & 2 & 0 & -3 \\
0 & 3 & -4 & -7 \\
0 & 0 & 1 & 1
\end{pmatrix}
\]
Example 3 — Back Solving

3. The simplified system is

\[-x_1 + 2x_2 = -3\]
\[3x_2 - 4x_3 = -7\]
\[x_3 = 1\]

4. Backsolve: \(x_3 = 1, x_2 = -1, \text{ and } x_1 = 1.\)
Elimination using MATLAB

- \( R_i \rightarrow R_i + aR_j \)
  - >> \( M(i,:) = M(i,:) + a*M(j,:) \)

- \( R_i \leftrightarrow R_j \)
  - >> \( M([i,j],:) = M([j,i],:) \)

- \( R_i \rightarrow aR_i \)
  - >> \( M(i,:) = a*M(i,:) \)
Example 4 \( Ax = b \)

\[
A = \begin{pmatrix}
1 & 2 & 5 & -1 \\
1 & 2 & -3 & 8 \\
3 & 6 & 7 & 6
\end{pmatrix}, \quad b = \begin{pmatrix}
-2 \\
-12 \\
-16
\end{pmatrix}
\]

1. Augmented matrix:

\[
M = [A, \ b] = \begin{pmatrix}
1 & 2 & 5 & -1 & -2 \\
1 & 2 & -3 & 8 & -12 \\
3 & 6 & 7 & 6 & -16
\end{pmatrix}
\]
Example 4 — Elimination

2. Elimination using MATLAB yields.

\[
\begin{pmatrix}
1 & 2 & 5 & -1 & -2 \\
0 & 0 & -8 & 9 & -10 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

♦ The yellow entries are the pivots.
3. Simplified system:

\[ x_1 + 2x_2 + 5x_3 - x_4 = -2 \]

\[ -8x_3 + 9x_4 = -10 \]

4. Backsolve:

- There are pivots in columns 1 & 3. These are pivot columns. \( x_1 \) and \( x_3 \) are called pivot variables.
- The other columns are called free columns. The variables \( x_2 \) and \( x_4 \) are called free variables.
The free variables may be assigned arbitrary values: 
\[ x_2 = s \text{ and } x_4 = t. \]

Backsolve for the pivot variables.

\[ x_3 = \frac{10 + 9x_4}{8} = \frac{5}{4} + \frac{9t}{8} \]
\[ x_1 = -2 - 2x_2 - 5x_3 + x_4 \]
\[ = -2 - 2s - 5\left(\frac{5}{4} + \frac{9t}{8}\right) + t \]
\[ = -\frac{33}{4} - 2s - \frac{37t}{8} \]
• The solutions are the vectors

\[ x = \begin{pmatrix} -\frac{33}{4} \\ 0 \\ \frac{5}{4} \\ 0 \end{pmatrix} + s \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t \begin{pmatrix} -\frac{37}{8} \\ 0 \\ \frac{9}{8} \\ 0 \end{pmatrix} \]

• The solution set is a plane in \( \mathbb{R}^4 \).
Method of Solution for $Ax = b$

There are four steps:

1. Use the augmented matrix $M = [A, \; b]$.
2. Use row operations to reduce the augmented matrix to row echelon form.
3. Write down the simplified system.
   - Assign arbitrary values to the free variables.
   - Backsolve for the pivot variables.