The Solution Set of $Ax = b$

Theorem: Let $x_p$ be a particular solution to $Ax_p = b$.

1. If $Ax_h = 0$ then $x = x_p + x_h$ also satisfies $Ax = b$.
2. If $Ax = b$, then there is a vector $x_h$ such that $Ax_h = 0$ and $x = x_p + x_h$.

Thus, the solution set for $Ax = b$ is known if we know one particular solution $x_p$ and the solution set for the homogeneous system $Ax_h = 0$.

The Nullspace of a Matrix

- The solution set for the homogeneous system $Ax = 0$ is called the nullspace of the matrix $A$. It is denoted by $\text{null}(A)$. Thus

  $\text{null}(A) = \{x \mid Ax = 0\}$.

- What are the properties of nullspaces?
- Is there a convenient way to describe them?
Properties of Nullspaces

Proposition: Let $A$ be a matrix.
1. If $x$ and $y$ are in $\text{null}(A)$, then $x + y$ is in $\text{null}(A)$.
2. If $a$ is a scalar and $x$ is in $\text{null}(A)$, then $ax$ is in $\text{null}(A)$.

Definition: A nonempty subset $V$ of $\mathbb{R}^n$ that has the properties
1. if $x$ and $y$ are vectors in $V$, then $x + y$ is in $V$,
2. if $a$ is a scalar, and $x$ is in $V$, then $ax$ is in $V$,
is called a subspace of $\mathbb{R}^n$.

Examples of Subspaces

- The nullspace of a matrix is a subspace.
- A line through 0, $V = \{tv | t \in \mathbb{R} \}$, is a subspace.
- A plane through 0, $V = \{av + bw | a, b \in \mathbb{R} \}$, is a subspace.
- $\{0\}$ and $\mathbb{R}^n$ are subspaces of $\mathbb{R}^n$.
  - These are called the trivial subspaces.

Linear Combinations

Proposition: Any linear combination of vectors in a subspace $V$ is also in $V$.

- Subspaces of $\mathbb{R}^n$ have the same linear structure as $\mathbb{R}^n$ itself.
- The nullspace of a matrix is a subspace, so it has the same linear structure as $\mathbb{R}^n$.
- The product of a matrix $A$ and a vector $x$ is the linear combination of the column vectors in $A$ with the elements of $x$ as coefficients.
Another Example of a Nullspace

\[
A = \begin{pmatrix}
4 & 3 & -1 \\
-3 & -2 & 1 \\
1 & 2 & 1
\end{pmatrix} \xrightarrow{rref} \begin{pmatrix}
1 & 0 & -1 \\
0 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

The nullspace of \(A\) is the set

\[\text{null}(A) = \{av \mid a \in \mathbb{R}\},\]

where \(v = (1, -1, 1)^T\).

- The nullspace of \(A\) consists of all multiples of \(v\).

Another Example of a Nullspace

\[
B = \begin{pmatrix}
4 & 3 & -1 & 6 \\
-3 & -2 & 1 & -4 \\
1 & 2 & 1 & 4
\end{pmatrix} \xrightarrow{rref} \begin{pmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 1 & 2 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

The nullspace of \(B\) is the set

\[\text{null}(B) = \{av + bw \mid a, b \in \mathbb{R}\},\]

where \(v = (1, -1, 1, 0)^T\) and \(w = (0, -2, 0, 1)^T\).

- \(\text{null}(B)\) consists of all linear combinations of \(v\) and \(w\).

The Span of a Set of Vectors

In every example the subspace has been the set of all linear combinations of a few vectors.

Definition: The span of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors \(v_1, v_2, \ldots, \) and \(v_k\) is denoted by

\[\text{span}(v_1, v_2, \ldots, v_k)\].

Proposition: If \(v_1, v_2, \ldots, \) and \(v_k\) are all vectors in \(\mathbb{R}^n\), then \(V = \text{span}(v_1, v_2, \ldots, v_k)\) is a subspace of \(\mathbb{R}^n\).
How Do We Know if $w \in \text{span}(v_1, v_2, \ldots, v_k)$?

1. Form the matrix $M = [v_1, v_2, \ldots, v_k]$ which has the vectors $v_1$, $v_2$, ..., and $v_k$ as its columns.
2. Solve the system $M a = w$.
   a. If there are no solutions, $w$ is NOT in $\text{span}(v_1, v_2, \ldots, v_k)$.
   b. If there is a solution $a = (a_1, a_2, \ldots, a_k)^T$, then
      
      $w = a_1 v_1 + a_2 v_2 + \ldots + a_k v_k$

      is in $\text{span}(v_1, v_2, \ldots, v_k)$.

Examples of Spans

Let $v_1 = (1, 2)^T$, $v_2 = (1, 0)^T$, and $v_3 = (2, 0)^T$.

- $\text{span}(v_1, v_2) = \mathbb{R}^2$. (Proof?)
- $\text{span}(v_1, v_3) = \mathbb{R}^2$. (Proof?)
- $\text{span}(v_2, v_3) = \text{span}(v_2)$. (Proof?)
  * $\text{span}(v_2, v_3) = \{t v_2 | t \in \mathbb{R} \}$.
  * $v_2$ and $v_3$ have the same direction.

Linear Independence of Two Vectors

We need a condition that will keep unneeded vectors out of a spanning list. We will work toward a general definition.

- Two vectors are \textit{linearly dependent} if one is a scalar multiple of the other.
- $v_2$ and $v_3$ are linearly dependent.
- $v_1$ and $v_2$ are \textit{linearly independent}. 
Linear Independence of Three Vectors

- Three vectors \( \mathbf{v}_1, \mathbf{v}_2, \) and \( \mathbf{v}_3 \) are linearly dependent if one is a linear combination of the other two.
- Example: \( \mathbf{v}_1 = (1, 0, 0)^T, \mathbf{v}_2 = (0, 1, 0)^T, \) and \( \mathbf{v}_3 = (1, 2, 0)^T. \) Notice that \( \mathbf{v}_3 = \mathbf{v}_1 + 2\mathbf{v}_2. \)
- Therefore \( \mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}. \)

Linear Independence

- Three vectors are linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.
- Non-trivial means that at least one of the coefficients is not 0.
- A set of vectors is linearly dependent if there is a non-trivial linear combination of them which equals the zero vector.

Definition: The vectors \( \mathbf{v}_1, \mathbf{v}_2, \ldots, \) and \( \mathbf{v}_k \) are linearly independent if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols, \( c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_k\mathbf{v}_k = \mathbf{0} \)
- \( \Rightarrow c_1 = c_2 = \cdots = c_k = 0. \)
When are $v_1$, $v_2$, $v_3$, and $v_k$ Linearly Independent?

1. Form the matrix $M = [v_1, v_2, \ldots, v_k]$ which has the vectors $v_1$, $v_2$, $\ldots$, and $v_k$ as its columns.
2. Find the nullspace, null($M$).
   a. If null($M$) = $\{0\}$, the vectors are linearly independent.
   b. If $a \in$ null($M$), and $a = (a_1, a_2, \ldots, a_k)^T \neq 0$, then
      $$a_1 v_1 + a_2 v_2 + \ldots + a_k v_k = 0$$
      and the vectors are linearly dependent.

Example 1

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 5 \\ 0 \\ -4 \\ 6 \end{pmatrix}
\end{align*}
\]

\[
M = [v_1, v_2, v_3] = \begin{pmatrix} 1 & -1 & 5 \\ -2 & -3 & 0 \\ 0 & 2 & -4 \\ 2 & 0 & 6 \end{pmatrix}
\]  
\[
\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\]

• null($M$) consists of all multiples of $a = (-3, 2, 1)^T$.
• $-3v_1 + 2v_2 + v_3 = 0$, so the vectors are linearly dependent.

Example 2

\[
\begin{align*}
v_1 &= \begin{pmatrix} 1 \\ -2 \\ 0 \\ 2 \end{pmatrix}, \\
v_2 &= \begin{pmatrix} -1 \\ -3 \\ 2 \\ 0 \end{pmatrix}, \\
v_3 &= \begin{pmatrix} 5 \\ 0 \\ -4 \\ 3 \end{pmatrix}
\end{align*}
\]

\[
M = [v_1, v_2, v_3] = \begin{pmatrix} 1 & -1 & 5 \\ -2 & -3 & 0 \\ 0 & 2 & -4 \\ 2 & 0 & 3 \end{pmatrix}
\]  
\[
\xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\]

• null($[v_1, v_2, v_3]$) = $\{0\}$.
• $v_1$, $v_2$, $v_3$ are linearly independent.
Basis of a Subspace
Definition: A set of vectors $v_1, v_2, \ldots, v_k$ form a basis of a subspace $V$ if
1. $V = \text{span}(v_1, v_2, \ldots, v_k)$
2. $v_1, v_2, \ldots, v_k$ are linearly independent.

Examples of Bases
• The vector $v = (1, -1, 1)^T$ is a basis for $\text{null}(A)$.
  • $\text{null}(A)$ is the subspace of $\mathbb{R}^3$ with basis $v$.
• The vectors $v = (1, -1, 1, 0)^T$ and $w = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
  • $\text{null}(B)$ is the subspace of $\mathbb{R}^4$ with basis $\{v, w\}$.

Existence of a Basis
Proposition: Let $V$ be a subspace of $\mathbb{R}^n$.
1. If $V \neq \{0\}$, then $V$ has a basis.
2. Bases are not unique, but every basis of $V$ has the same number of elements.
Definition: The dimension of a subspace $V$ is the number of elements in a basis of $V$. 
Another Example of a Nullspace

\[ A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix} \] 

\[ \text{rref}(A) \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

- \( \text{null}(A) \) is the subspace of \( \mathbb{R}^4 \) with basis \((1,1,0,0)^T\) and \((0,0,1,-1)^T\).
- \( \text{null}(A) \) has dimension 2.
- In MATLAB, use commands \( \text{null}(A) \) or \( \text{null}(A,'r') \).

Product of a Matrix with a Vector

- The product of a matrix \( A \) and a vector \( x \) is the linear combination of the columns of \( A \) with the elements of \( x \) as coefficients.
- Example:

\[ \begin{pmatrix} 3 & -4 & 5 \\ -1 & 2 & -2 \end{pmatrix} \begin{pmatrix} 13 \\ -5 \\ 23 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} + (-5) \begin{pmatrix} -4 \\ 2 \end{pmatrix} + 23 \begin{pmatrix} 5 \\ -2 \end{pmatrix} \]