Basis of a Subspace

Definition: A set of vectors $v_1, v_2, \ldots, v_k$ form a basis of a subspace $V$ if

1. $V = \text{span}(v_1, v_2, \ldots, v_k)$
2. $v_1, v_2, \ldots, v_k$ are linearly independent.

- The best way to describe a subspace is to give a basis.

Examples of Bases

- The vector $v = (1, -1, 1)^T$ is a basis for $\text{null}(A)$.
- $\text{null}(A)$ is the subspace of $\mathbb{R}^3$ with basis $v$.
- The vectors $v = (1, -1, 1, 0)^T$ and $w = (0, -2, 0, 1)^T$ form a basis for $\text{null}(B)$.
- $\text{null}(B)$ is the subspace of $\mathbb{R}^4$ with basis $\{v, w\}$. 
Existence of a Basis

Proposition: Let $V$ be a subspace of $\mathbb{R}^n$.

1. If $V \neq \{0\}$, then $V$ has a basis.

2. Bases are not unique, but every basis of $V$ has the same number of elements.

Definition: The dimension of a subspace $V$ is the number of elements in a basis of $V$.

Another Example of a Nullspace

$A = \begin{pmatrix} 3 & -3 & 1 & -1 \\ -2 & 2 & -1 & 1 \\ 1 & -1 & 0 & 0 \\ 13 & -13 & 5 & -5 \end{pmatrix}$

$rref(A) \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

- $\text{null}(A)$ is the subspace of $\mathbb{R}^4$ with basis $(1,1,0,0)^T$ and $(0,0,1,1)^T$.
- $\text{null}(A)$ has dimension 2.
- In MATLAB, use commands `null(A)` or `null(A,'r')`.

Nonsingular Matrices

Let $A$ be an $n \times n$ matrix. We know the following:

- $A$ is nonsingular if the equation $Ax = b$ has a solution for any right hand side $b$. (This is the definition.)
- If $A$ is nonsingular then $Ax = b$ has a unique solution for any right hand side $b$.
- $A$ is singular if and only if the homogeneous equation $Ax = 0$ has a non-zero solution.
- $\text{null}(A)$ is non-trivial $\Leftrightarrow A$ is singular.
Determinants in 2D

- How do we decide if a matrix $A$ is nonsingular?
- $A$ is nonsingular if and only if when put into row echelon form, the matrix has nonzero entries along the diagonal.
- Example: the general $2 \times 2$ matrix
  \[
  A = \begin{pmatrix}
  a & b \\
  c & d
  \end{pmatrix}
  \]
  is nonsingular if and only if $ad - bc \neq 0$.
- We define $ad - bc$ to be the determinant of $A$.

Determinants in 3D

- The same (but more difficult) argument shows that $A$ is nonsingular if and only if
  \[
  a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} \neq 0.
  \]
- This will be the determinant of $A$.

Main Theorem

We will define the determinant of a square matrix $A$ so that the next theorem is true.

Theorem: The $n \times n$ matrix $A$ is nonsingular if and only if $\det(A) \neq 0$.

Corollary: If $A$ is an $n \times n$ matrix, then $\text{null}(A)$ contains a nonzero vector if and only if $\det(A) = 0$.

- The corollary contains the most important fact about determinants for ODEs.
Matrices and Minors

The general \( n \times n \) matrix has the form

\[
A = \begin{pmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
a_{21} & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix}
\]

Definition: The \( ij \)-minor of an \( n \times n \) matrix \( A \) is the \((n-1) \times (n-1)\) matrix \( A_{ij} \) obtained from \( A \) by deleting the \( i \)th row and the \( j \)th column.

Definition of Determinant

Definition: The determinant of an \( n \times n \) matrix \( A \) is defined to be

\[
det(A) = \sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j}).
\]

- The definition is inductive.
- It assumes we know how to compute the determinants of \((n-1) \times (n-1)\) matrices.
- We start with the \( 2 \times 2 \) matrix.

Example

\[
det \begin{pmatrix}
2 & 1 & 0 \\
3 & -2 & 4 \\
-1 & 5 & 3
\end{pmatrix} = (-1)^2 \times 2 \times det \begin{pmatrix}
-2 & 4 \\
5 & 3
\end{pmatrix} + (-1)^3 \times 1 \times det \begin{pmatrix}
3 & 4 \\
-1 & 3
\end{pmatrix} = 2 \times (-26) - 13 = -65
\]
Expansion by the $i^{th}$ Row

For any $i$, we have

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}).$$

- This is called expansion by the $i^{th}$ row.

- Example:

$$\det \begin{pmatrix} 5 & -6 & 3 \\ 0 & 4 & 0 \\ 2 & -1 & 9 \end{pmatrix} = 4 \cdot \det \begin{pmatrix} 5 & 3 \\ 2 & 9 \end{pmatrix} = 156.$$

Properties of the Determinant

- The formula for the determinant of a matrix $A$ is the sum of $n!$ products of the entries of $A$ (sometimes $\times -1$).

- Each summand is the product of $n$ entries, one from each row, and one from each column.

- The determinant of a triangular matrix is the product of the diagonal terms.

- We can use row operations to compute determinants.

Row Operations and Determinants

If $B$ is obtained from $A$ by

- adding a multiple of one row to another, then

$$\det(B) = \det(A).$$

- interchanging two rows, then

$$\det(B) = -\det(A).$$

- multiplying a row by $c \neq 0$, then

$$\det(B) = c \det(A).$$
Example

\[
A = \begin{pmatrix}
-5 & 2 & 3 \\
25 & -9 & -12 \\
10 & 7 & 17 \\
\end{pmatrix}
\]

\[\text{det}(A) = 50\]

More Properties

• If A has two equal rows, then \(\text{det}(A) = 0\).
• If A has a row of all zeros, then \(\text{det}(A) = 0\).
• \(\text{det}(A^T) = \text{det}(A)\).
• If A has two equal columns, then \(\text{det}(A) = 0\).
• If A has a column of all zeros, then \(\text{det}(A) = 0\).

Column Operations and Determinants

If B is obtained from A by
• adding a multiple of one column to another, then
  \[\text{det}(B) = \text{det}(A)\].
• interchanging two columns, then
  \[\text{det}(B) = -\text{det}(A)\].
• multiplying a column by \(c \neq 0\), then
  \[\text{det}(B) = c\text{det}(A)\].
Expansion by a Column

We can also expand by a column.

\[ \det(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}). \]

- This is called expansion by the \( j \)th column.

Example

\[ A = \begin{pmatrix} -5 & -6 & 0 \\ 3 & 4 & 0 \\ -8 & -16 & 9 \end{pmatrix} \]

\[
\det(A) = 9 \cdot \det \begin{pmatrix} -5 & -6 \\ 3 & 4 \end{pmatrix} \\
= 9 \cdot (-2) \\
= -18
\]

Determinants and Bases

Proposition: A collection of \( n \) vectors \( v_1, v_2, \ldots, v_n \) in \( \mathbb{R}^n \) is a basis for \( \mathbb{R}^n \) if and only if

\[ \det([v_1 \ v_2 \ \ldots \ v_n]) \neq 0. \]
Examples
\[
\det \begin{pmatrix}
1 & 1 & 1 & 1 \\
2 & 1 & -1 & -2 \\
-2 & -1 & 1 & 1 \\
2 & 2 & 1 & 1 \\
\end{pmatrix} = 1.
\]
\[
\det \begin{pmatrix}
3 & -1 & 0 & 1 \\
12 & -6 & 0 & 5 \\
32 & -15 & -3 & 13 \\
18 & -10 & -1 & 8 \\
\end{pmatrix} = -1.
\]

The Span of a Set of Vectors

Definition: The span of a set of vectors is the set of all linear combinations of those vectors. The span of the vectors \(v_1, v_2, \ldots, v_k\) is denoted by
\[
\text{span}(v_1, v_2, \ldots, v_k).
\]

Linear Independence

Definition: The vectors \(v_1, v_2, \ldots, v_k\) are linearly independent if the only linear combination of them which is equal to the zero vector is the one with all of the coefficients equal to 0.

- In symbols,
\[
c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0
\]
\[
\Rightarrow c_1 = c_2 = \cdots = c_k = 0.
\]
**Example of a Nullspace**

\[ A = \begin{pmatrix} 4 & 3 & -1 \\ -3 & -2 & 1 \\ 1 & 2 & 1 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \]

The nullspace of \( A \) is the set

\[ \text{null}(A) = \{av \mid a \in \mathbb{R} \}, \]

where \( v = (1, -1, 1)^T \).

- The nullspace of \( A \) consists of all multiples of \( v \).

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**Another Example of a Nullspace**

\[ B = \begin{pmatrix} 4 & 3 & -1 & 6 \\ -3 & -2 & 1 & -4 \\ 1 & 2 & 1 & 4 \end{pmatrix} \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \]

The nullspace of \( B \) is the set

\[ \text{null}(B) = \{av + bw \mid a, b \in \mathbb{R} \}, \]

where \( v = (1, -1, 1, 0)^T \) and \( w = (0, -2, 0, 1)^T \).

- \( \text{null}(B) \) consists of all linear combinations of \( v \) and \( w \).