Math 211

Lecture #22

Systems of ODEs

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Systems of Differential Equations

Example: A mixed population of predators (foxes) and prey (rabbits).

- The prey, $x(t)$, flourish in the absence of the predators.
- The predators, $y(t)$, depend on the prey as a food source, and would die out in the absence of the prey.
- For predation to take place there must be an encounter between a predator and a prey.
Predator-Prey Model

The basic model is $x' = r_x \cdot x$ and $y' = r_y \cdot y$, where $r_x$ and $r_y$ are the reproductive rates.

- $r_x = a > 0$ if $y = 0$, and decreases as $y$ increases.
  - $r_x = a - by$.
- $r_y = -c < 0$ if $x = 0$, and increases as $x$ increases.
  - $r_y = -c + dx$.
- The system becomes: $x' = (a - by)x$
  
  $y' = (-c + dx)y$

- MATLAB & pplane6.
General System in 2D

\[ x' = f(t, x, y) \]
\[ y' = g(t, x, y) \]

- Example:
  \[ x' = y \]
  \[ y' = -x \]

- Solution: \( x(t) = \sin t \) and \( y(t) = \cos t \)
  
  ✦ Verify by direct substitution.
General System in Higher D

\[ x_1' = f_1(t, x_1, x_2, \ldots, x_n) \]
\[ x_2' = f_2(t, x_1, x_2, \ldots, x_n) \]
\[ \vdots \]
\[ x_n' = f_n(t, x_1, x_2, \ldots, x_n) \]

- The *dimension* of a system is the number of unknown functions = the number of equations.
  - The *predator-prey model* has dimension 2.
Vector Notation — 2D

- In 2D set $u_1(t) = x(t)$ & $u_2(t) = y(t)$, and
  
  $$
  u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.
  $$

- Then in the example

  $$
  x' = y \quad \iff \quad u' = \begin{pmatrix} u'_1 \\ u'_2 \end{pmatrix} = \begin{pmatrix} u_2 \\ -u_1 \end{pmatrix}
  $$
Vector Notation — Planar System

- For the **general case** use **vector** notation and set

\[
F(t, \mathbf{u}) = \begin{pmatrix}
    f(t, u_1, u_2) \\
    g(t, u_1, u_2)
\end{pmatrix}.
\]

- Then

\[
x' = f(t, x, y) \quad \Leftrightarrow \quad u' = F(t, \mathbf{u})
\]

\[
y' = g(t, x, y)
\]
Vector Notation — General

- In higher dimensions, set

\[
\mathbf{x}(t) = \begin{pmatrix}
    x_1(t) \\
    x_2(t) \\
    \vdots \\
    x_n(t)
\end{pmatrix}
\quad \mathbf{f}(t, \mathbf{x}) = \begin{pmatrix}
    f_1(t, \mathbf{x}) \\
    f_2(t, \mathbf{x}) \\
    \vdots \\
    f_n(t, \mathbf{x})
\end{pmatrix}.
\]

- The general system can be written

\[
\mathbf{x}' = \mathbf{f}(t, \mathbf{x}).
\]
Vector Notation — Predator-Prey Model

For the predator-prey model set $u_1 = x$, and $u_2 = y$.
Then the system can be written

$$\mathbf{u}' = \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} (a - bu_2)u_1 \\ (-c + bu_1)u_2 \end{pmatrix}.$$ 

- This is an *autonomous* system.
  - The RHS has no explicit dependence on $t$. 

Initial Value Problem

\[ x' = f(t, x) \quad x(t_0) = x_0. \]

- Each component of \( x(t_0) \) must be specified.

- Example:

\[
\begin{align*}
x' &= y \\
y' &= -x
\end{align*}
\]

with

\[
\begin{align*}
x(0) &= 2 \\
y(0) &= 13
\end{align*}
\]

- PP model: Both the initial prey population and the initial predator population must be specified.
Reduction of Higher Order Equation to a System

For any higher order equation there is a first order system which is equivalent to it, in the sense that solutions of the system lead easily to solutions of the equation, and vice versa.

- Reduces the study of higher order equations to the study of systems
- Useful for the computation of solutions of higher order equations.
Example of Reduction

- Third-order equation: \( y''' + 2yy' = 3 \cos t \)
- Set \( x_1 = y, \ x_2 = y', \) and \( x_3 = y''. \)
- Then

\[
\begin{align*}
x_1' &= x_2 \\
x_2' &= x_3 \\
x_3' &= 3 \cos t - 2x_1x_2
\end{align*}
\]

- This system is not autonomous.
Geometric Interpretation of Solutions

- Parametric plot
  - Tangent vectors
- Vector fields
- Phase plane
- pplane6 for planar autonomous systems.