Math 211

Lecture #24

Linear Systems of ODEs

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General Linear Systems

\[ x'_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + f_1 \]
\[ x'_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + f_2 \]
\[ \vdots = \vdots \]
\[ x'_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + f_n \]

- The coefficients can depend on \( t \).
• Set

\[ x = (x_1, x_2, \ldots, x_n)^T \]

\[ f(t) = (f_1(t), f_2(t), \ldots, f_n(t))^T \]

\[ A = \begin{pmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} \]

• The system becomes \( x' = Ax + f \).
Homogeneous Systems

An *homogeneous* system is one of the form

\[ x' = Ax \]

**Proposition:** Suppose that \( x_1(t), x_2(t), \ldots, \) and \( x_k(t) \) are solutions to the homogeneous system \( x' = Ax \), and \( c_1, c_2, \ldots, \) and \( c_k \) are scalars. Then

\[ x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_k x_k(t) \]

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.
Very Important Example

• The system

\[ x' = Ax \quad \text{with} \quad A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

has solutions

\[ x_1(t) = e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad x_2(t) = e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}. \]

♦ Verify by direct substitution.

• Proposition \( \Rightarrow x(t) = C_1x_1(t) + C_2x_2(t) \) is a solution for any constants \( C_1 \) and \( C_2 \).

♦ Is this the general solution?
• Let \( y \) be a solution of \( y' = Ay \). Can we find \( C_1 \) and \( C_2 \) so that \( y(t) = C_1x_1(t) + C_2x_2(t) \) for all \( t \)?

• Let’s ask a simpler question. Can we find \( C_1 \) and \( C_2 \) so that \( y(0) = C_1x_1(0) + C_2x_2(0) \)?
  
  ♦ Yes, since \( x_1(0) \) and \( x_2(0) \) are linearly independent.

• Uniqueness theorem \( \Rightarrow \)

\[
y(t) = C_1x_1(t) + C_2x_2(t) \quad \text{for all } t.
\]

• Thus, every solution to \( x' = Ax \) is a linear combination of \( x_1 \) and \( x_2 \).

• Can we generalize this result?
Key Point in the Argument

- Need to solve the equation

\[ y_0 = C_1 x_1(0) + C_2 x_2(0) \]

for any \( y_0 = y(0) \).

- Possible if \( x_1(0) \) and \( x_2(0) \) are linearly independent.

- Uniqueness then implies that

\[ y(t) = C_1 x_1(t) + C_2 x_2(t) \quad \text{for all } t \]

- We needed \( x_1(t) \) and \( x_2(t) \) to be linearly independent at only one point.
**Proposition:** $x_1(t), x_2(t), \ldots , \text{and } x_k(t)$ solutions to the homogeneous system $x' = Ax$ on the interval $I$.

1. If $x_1(t_0), x_2(t_0), \ldots , \text{and } x_k(t_0)$ are linearly independent for some $t_0 \in I$, then they are linearly independent for all $t \in I$.

2. If $x_1(t_0), x_2(t_0), \ldots , \text{and } x_k(t_0)$ are linearly dependent for some $t_0 \in I$, then they are linearly dependent for all $t \in I$. 
Linear Independence

Definition: A set of \( k \) solutions to the linear system \( x' = Ax \) is *linearly independent* if they are linearly independent at one value of \( t \).

- Proposition \( \implies \) the solutions are linearly independent for all values of \( t \).
Structure of the Solution Space

**Theorem:** Suppose that $x_1(t)$, $x_2(t)$, ..., and $x_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $x' = Ax$ on the interval $I$. Then every solution is a linear combination of $x_1(t)$, $x_2(t)$, ..., and $x_n(t)$.

- That is, if $x(t)$ is any solution, then there are constants $C_1$, $C_2$, ..., and $C_n$ such that

$$x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_nx_n(t).$$

- The general solution is a linear combination of $x_1(t)$, $x_2(t)$, ..., and $x_n(t)$. 
Solution Strategy

- The obvious strategy for completely solving an $n \times n$ homogeneous system is to look for $n$ linearly independent solutions.

**Definition:** A set of $n$ linear independent solutions to the $n \times n$ homogeneous system $x' = Ax$ is called a fundamental set of solutions.

- We will develop methods of finding fundamental sets of solutions.
Examples: \( \mathbf{x}' = A \mathbf{x} \)

- **Example 1:** \( A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \)
  
  \( \mathbf{x}_1(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix} \) and \( \mathbf{x}_2(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} \)
  
  is a fundamental set of solutions.

- **Example 2:** \( A = \begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix} \)
  
  \( \mathbf{x}_1(t) = e^t \begin{pmatrix} -2 \\ 1 \end{pmatrix} \) and \( \mathbf{x}_2(t) = e^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \)
  
  is a fundamental set of solutions.
Linear Systems with Constant Coefficients

- We will solve \textit{homogeneous} systems, \( x' = Ax \), first.
- We will be able to find explicit solutions.
- To motivate what we do, we will start with the easiest case, dimension = 1.
  - One equation: \( x' = ax \), where \( a \) is a constant.
  - Solution: \( x(t) = Ce^{at} \)
  - All solutions are exponentials. Can we find exponential solutions to a system of equations?
Exponential Solutions to $x' = Ax$

- Look for solution of the form $x(t) = e^{\lambda t} \mathbf{v}$, where $\mathbf{v}$ is a vector with constant entries.

- Substituting we get

  $x' = \lambda e^{\lambda t} \mathbf{v}$

  $Ax = e^{\lambda t} A\mathbf{v}$

- Hence $x' = Ax \iff A\mathbf{v} = \lambda \mathbf{v}$

- If $A\mathbf{v} = \lambda \mathbf{v}$ then $x(t) = e^{\lambda t} \mathbf{v}$ is a solution.

- Can we find $\lambda$ and $\mathbf{v}$ such that $A\mathbf{v} = \lambda \mathbf{v}$?
Definition: \( \lambda \) is an eigenvalue of \( A \) if there is a nonzero vector \( v \) such that \( Av = \lambda v \). If \( \lambda \) is an eigenvalue of \( A \), then any vector \( v \) such that \( Av = \lambda v \) is called an eigenvector associated with \( \lambda \).

- If \( \lambda \) an eigenvalue of \( A \), and \( v \) is an associated nonzero eigenvector, then \( x(t) = e^{\lambda t}v \) is a solution to \( x' = Ax \).
  
  ♦ Thus we have a way to find some solutions to systems with constant coefficients.

- How do we find eigenvalues and eigenvectors?
Finding Eigenvalues

$\lambda$ is an \textit{eigenvalue} of $A$

$\iff$ there is a vector $\mathbf{v} \neq \mathbf{0}$ such that $A\mathbf{v} = \lambda \mathbf{v}$.

$\iff \mathbf{v} \neq \mathbf{0}$ and $0 = A\mathbf{v} - \lambda \mathbf{v}$

$= A\mathbf{v} - \lambda I \mathbf{v}$

$= (A - \lambda I)\mathbf{v}$

$\iff A - \lambda I$ has a nontrivial nullspace.

$\iff \det(A - \lambda I) = 0.$
Example

\[ A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

\[ A - \lambda I = \begin{pmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{pmatrix} \]

\[ \det(A - \lambda I) = (-4 - \lambda)(1 - \lambda) + 6 \]

\[ = \lambda^2 + 3\lambda + 2 \]

\[ = (\lambda + 1)(\lambda + 2) \]

- \( A \) has eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \).
The Characteristic Polynomial

\[ A = \begin{pmatrix}
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn}
\end{pmatrix} \]

\[ A - \lambda I = \begin{pmatrix}
    a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda
\end{pmatrix} \]
• If $A$ is an $n \times n$ matrix $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree $n$.

**Definition:** The *characteristic polynomial* of the $n \times n$ matrix $A$ is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

• Thus, the *eigenvalues* of $A$ are the roots of the characteristic equation.
Our Solution Strategy for $x' = Ax$

If $A$ is $n \times n$, we are looking for $n$ linearly independent solutions.

- Each eigenvalue $\lambda$ of $A$ has by definition an associated nonzero eigenvector $v$. This gives us the solution, $x(t) = e^{\lambda t}v$.

- The eigenvalues of $A$ are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$.
  - $p(\lambda)$ has degree $n$, and usually has $n$ roots.

- Therefore, there are usually $n$ different solutions.
  - Are they linearly independent?
Finding Eigenvectors

\( \mathbf{v} \) is an eigenvector associated with the eigenvalue \( \lambda \) if

\[
A\mathbf{v} = \lambda \mathbf{v}
\]

\[\Leftrightarrow (A - \lambda I)\mathbf{v} = 0\]

\[\Leftrightarrow \mathbf{v} \in \text{null}(A - \lambda I)\]

- The set of all eigenvectors associated to the eigenvalue \( \lambda \) is equal to the nullspace of \( A - \lambda I \).
  - It is a subspace of \( \mathbb{R}^n \).
  - It is called the eigenspace of \( \lambda \).
Example: \[ A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

\( A \) has eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \).

- \( \lambda_1 = -1: \) \( A - \lambda_1 I = \begin{pmatrix} -4 + 1 & 2 \\ -3 & 1 + 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} \)
  - \( v_1 = (2, 3)^T \) is an eigenvector.
  - \( x_1(t) = e^{\lambda_1 t}v_1 = e^{-t}(2, 3)^T \) is a solution.

- \( \lambda_2 = -2: \) \( A - \lambda_2 I = \begin{pmatrix} -4 + 2 & 2 \\ -3 & 1 + 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} \)
  - \( v_2 = (1, 1)^T \) is an eigenvector.
  - \( x_2(t) = e^{\lambda_2 t}v_2 = e^{-2t}(1, 1)^T \) is a solution.
Example (cont.)

• $x_1(0) = v_1$ and $x_2(0) = v_2$ are linearly independent.

• $x_1$ and $x_2$ form a fundamental set of solutions.

• The general solution is the set of all linear combinations:

$$x(t) = C_1 x_1(t) + C_2 x_2(t)$$

$$= C_1 e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2C_1 e^{-t} + C_2 e^{-2t} \\ 3C_1 e^{-t} + C_2 e^{-2t} \end{pmatrix}$$
Procedure to Solve $x' = Ax$

- Find the **eigenvalues** of $A$, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue $\lambda$ find the **eigenspace**, which is equal to $\text{null}(A - \lambda I)$.
- If $\lambda$ is an eigenvalue and $v$ is an associated nonzero eigenvector, $x(t) = e^{\lambda t}v$ is a **solution**.
- Show that $n$ of these are linearly independent, *if we can*.
  - ♦ This must be explored further.