Math 211

Lecture #25

Exponential Solutions

October 25, 2002
Homogeneous Systems

- These are systems of the form

  \[ x' = Ax, \]

  where \( A \) is an \( n \times n \) matrix.

- We are looking primarily at homogeneous systems with constant coefficients.
Structure of the Solution Space

Theorem: Suppose that $x_1(t), x_2(t), \ldots, \text{and } x_n(t)$ are linearly independent solutions to the $n \times n$ homogeneous system $x' = Ax$ on the interval $I$. Then every solution is a linear combination of $x_1(t), x_2(t), \ldots, \text{and } x_n(t)$.

That is, if $x(t)$ is a solution, then there are constants $C_1, C_2, \ldots, \text{and } C_n$ such that

$$x(t) = C_1 x_1(t) + C_2 x_2(t) + \cdots + C_n x_n(t).$$
Solution Strategy

- The obvious strategy for completely solving the system is to look for \( n \) linearly independent solutions.

Definition: A set of \( n \) linear independent solutions to the \( n \times n \) homogeneous system \( x' = Ax \) is called a fundamental set of solutions.

- We will look for fundamental sets of solutions.
Exponential Solutions to $\mathbf{x}' = A\mathbf{x}$

- Look for solution of the form $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$, where $\mathbf{v}$ is a vector with constant entries.

- Substituting we get

$$\begin{align*}
\mathbf{x}' &= \lambda e^{\lambda t} \mathbf{v} \\
A\mathbf{x} &= e^{\lambda t} A\mathbf{v}
\end{align*}$$

- Hence $\mathbf{x}' = A\mathbf{x} \iff A\mathbf{v} = \lambda \mathbf{v}$

- If $A\mathbf{v} = \lambda \mathbf{v}$ then $\mathbf{x}(t) = e^{\lambda t} \mathbf{v}$ is a solution.

- Can we find $\lambda$ and $\mathbf{v}$ such that $A\mathbf{v} = \lambda \mathbf{v}$?
Definition: \( \lambda \) is an eigenvalue of \( A \) if there is a nonzero vector \( v \) such that \( Av = \lambda v \). If \( \lambda \) is an eigenvalue of \( A \), then any vector \( v \) such that \( Av = \lambda v \) is called an eigenvector associated with \( \lambda \).

- If \( \lambda \) an eigenvalue of \( A \), and \( v \) is an associated nonzero eigenvector, then \( x(t) = e^{\lambda t}v \) is a solution to \( x' = Ax \).
  
  ♦ Thus we have a way to find some solutions to systems with constant coefficients.

- How do we find eigenvalues and eigenvectors?
Finding Eigenvalues

\( \lambda \) is an eigenvalue of \( A \)

\( \iff \) there is a vector \( v \neq 0 \) such that \( Av = \lambda v \).

\( \iff \) \( v \neq 0 \) and \( 0 = Av - \lambda v \)

\[ = Av - \lambda I v \]

\[ = (A - \lambda I)v \]

\( \iff \) \( A - \lambda I \) has a nontrivial nullspace.

\( \iff \) \( \text{det}(A - \lambda I) = 0 \).
Example

\[ A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \]

\[ A - \lambda I = \begin{pmatrix} -4 - \lambda & 2 \\ -3 & 1 - \lambda \end{pmatrix} \]

\[ \det(A - \lambda I) = (-4 - \lambda)(1 - \lambda) + 6 \]

\[ = \lambda^2 + 3\lambda + 2 \]

\[ = (\lambda + 1)(\lambda + 2) \]

- \( A \) has eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \).
The Characteristic Polynomial

\[ A = \begin{pmatrix} 
    a_{11} & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} 
\end{pmatrix} \]

\[ A - \lambda I = \begin{pmatrix} 
    a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\
    a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda 
\end{pmatrix} \]
• If $A$ is an $n \times n$ matrix $p(\lambda) = \det(A - \lambda I)$ is a polynomial of degree $n$.

**Definition:** The *characteristic polynomial* of the $n \times n$ matrix $A$ is

$$p(\lambda) = \det(A - \lambda I).$$

The *characteristic equation* is $p(\lambda) = 0$.

• Thus, the *eigenvalues* of $A$ are the roots of the characteristic equation.
Our Solution Strategy for $x' = Ax$

If $A$ is $n \times n$, we are looking for $n$ linearly independent solutions.

- Each eigenvalue $\lambda$ of $A$ has by definition an associated nonzero eigenvector $v$. This gives us the solution, $x(t) = e^{\lambda t} v$.

- The eigenvalues of $A$ are the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I) = 0$.
  - $p(\lambda)$ has degree $n$, and usually has $n$ roots.

- Therefore, there are usually $n$ different solutions.
  - Are they linearly independent?
Finding Eigenvectors

\( \mathbf{v} \) is an eigenvector associated with the eigenvalue \( \lambda \) if

\[
A\mathbf{v} = \lambda \mathbf{v}
\]

\[\Leftrightarrow (A - \lambda I)\mathbf{v} = \mathbf{0}\]

\[\Leftrightarrow \mathbf{v} \in \text{null}(A - \lambda I)\]

- The set of all eigenvectors associated to the eigenvalue \( \lambda \) is equal to the nullspace of \( A - \lambda I \).
  - It is a subspace of \( \mathbb{R}^n \).
  - It is called the eigenspace of \( \lambda \).
Example: \( A = \begin{pmatrix} -4 & 2 \\ -3 & 1 \end{pmatrix} \)

\( A \) has eigenvalues \( \lambda_1 = -1 \) and \( \lambda_2 = -2 \).

- \( \lambda_1 = -1 \): \( A - \lambda_1 I = \begin{pmatrix} -4 + 1 & 2 \\ -3 & 1 + 1 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ -3 & 2 \end{pmatrix} \)
  
  \( \mathbf{v}_1 = (2, 3)^T \) is an eigenvector.

  \( \mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-t} (2, 3)^T \) is a solution.

- \( \lambda_2 = -2 \): \( A - \lambda_2 I = \begin{pmatrix} -4 + 2 & 2 \\ -3 & 1 + 2 \end{pmatrix} = \begin{pmatrix} -2 & 2 \\ -3 & 3 \end{pmatrix} \)

  \( \mathbf{v}_2 = (1, 1)^T \) is an eigenvector.

  \( \mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2 = e^{-2t} (1, 1)^T \) is a solution.
Example (cont.)

- \( x_1(0) = v_1 \) and \( x_2(0) = v_2 \) are linearly independent.
- \( x_1 \) and \( x_2 \) form a fundamental set of solutions.
- The general solution is the set of all linear combinations:

\[
x(t) = C_1 x_1(t) + C_2 x_2(t)
\]

\[
= C_1 e^{-t} \begin{pmatrix} 2 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 1 \\ 1 \end{pmatrix}
\]

\[
= \begin{pmatrix} 2C_1 e^{-t} + C_2 e^{-2t} \\ 3C_1 e^{-t} + C_2 e^{-2t} \end{pmatrix}
\]
Procedure to Solve $x' = Ax$

- Find the eigenvalues of $A$, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue $\lambda$ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If $\lambda$ is an eigenvalue and $v$ is an associated nonzero eigenvector, $x(t) = e^{\lambda t}v$ is a solution.
- Show that $n$ of these are linearly independent, if we can.

- This must be explored further.
Solving $x' = Ax$

Cases to be Considered

- Distinct real eigenvalues.
  - In this case the method works as described.

- Complex eigenvalues.
  - The method yields complex solutions, but we will want real solutions.

- Repeated eigenvalues.
  - The method does not always give enough solutions.
    - This is the hard case.
Planar System $x' = Ax$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

In nonvector form

$x'_1 = a_{11} x_1 + a_{12} x_2$

$x'_2 = a_{21} x_1 + a_{22} x_2$
The Characteristic Polynomial

\[ p(\lambda) = \det(A - \lambda I) \]

\[ = \det \begin{pmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{pmatrix} \]

\[ = \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}a_{22} - a_{12}a_{21}) \]

\[ = \lambda^2 - T\lambda + D, \]

where

- \( D = a_{11}a_{22} - a_{12}a_{21} = \det(A) \)
- \( T = a_{11} + a_{22} = \text{tr}(A) \) is the trace of \( A \).

- The trace of a matrix is the sum of its diagonal elements.
The Eigenvalues of $A$

- The eigenvalues of $A$ are the roots of the characteristic equation $p(\lambda) = \lambda^2 - T\lambda + D = 0$.

\[ \lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}. \]

- Three cases:
  - 2 distinct real roots if $T^2 - 4D > 0$
  - 2 complex conjugate roots if $T^2 - 4D < 0$
  - Double real root if $T^2 - 4D = 0$
Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

**Proposition:** Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix $A$, and that $v_1 \neq 0$ and $v_2 \neq 0$ are eigenvectors associated with $\lambda_1$ and $\lambda_2$, respectively. Then $v_1$ and $v_2$ are linearly independent.
Two Distinct Real Eigenvalues

\[ \lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}, \quad \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2} \]

- \( T^2 - 4D > 0 \) so \( \lambda_1 < \lambda_2 \).
- There are associated nonzero eigenvectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \).
- Solutions \( \mathbf{x}_1(t) = e^{\lambda_1 t} \mathbf{v}_1 \) and \( \mathbf{x}_2(t) = e^{\lambda_2 t} \mathbf{v}_2 \).
- \( \mathbf{x}_1(0) = \mathbf{v}_1 \) and \( \mathbf{x}_2(0) = \mathbf{v}_2 \) are linearly independent; \( \mathbf{x}_1(t) \) and \( \mathbf{x}_2(t) \) form a fundamental set of solutions.
- The general solution is \( \mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2 \).
Example

\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} -6 & -8 \\ 4 & 6 \end{pmatrix} \]

- **Characteristic polynomial**: \( p(\lambda) = \lambda^2 - 4 \).

- **Eigenvalues**: \( \lambda_1 = -2 \) and \( \lambda_2 = 2 \).
  
  - \( \lambda_1 = -2 \). **Eigenvector**: \( \mathbf{v}_1 = (-2, 1)^T \).
    
    \( \quad \Rightarrow \) **Solution**: \( x_1(t) = e^{\lambda_1 t} \mathbf{v}_1 = e^{-2t}(-2, 1)^T \).

  - \( \lambda_2 = 2 \). **Eigenvector**: \( \mathbf{v}_2 = (-1, 1)^T \).
    
    \( \quad \Rightarrow \) **Solution**: \( x_2(t) = e^{\lambda_2 t} \mathbf{v}_2 = e^{2t}(-1, 1)^T \).
• \( x_1 \) and \( x_2 \) are a **fundamental set** of solutions.

• The general solution is

\[
x(t) = C_1 x_1(t) + C_2 x_2(t)
\]

\[
= C_1 e^{-2t} \begin{pmatrix} -2 \\ 1 \end{pmatrix} + C_2 e^{2t} \begin{pmatrix} -1 \\ 1 \end{pmatrix}.
\]
Initial Value Problem

Solve $x' = Ax$ with the initial condition $x(0) = (1, 4)^T$.

- We need

$$x(0) = C_1 x_1(0) + C_2 x_2(0)$$

- $C_1 = -5$ and $C_2 = 9$.

- The solution is

$$x(t) = -5x_1(t) + 9x_2(t) = \begin{pmatrix} 10e^{-2t} - 9e^{2t} \\ -5e^{-2t} + 9e^{2t} \end{pmatrix}.$$
Homogeneous Systems

\[ x' = Ax \]

**Proposition:** Suppose that \( x_1(t) \), \( x_2(t) \), \( \ldots \), and \( x_k(t) \) are solutions to the homogeneous system, and \( c_1 \), \( c_2 \), \( \ldots \), and \( c_k \) are scalars. Then

\[ x(t) = c_1 x_1(t) + c_2 x_2(t) + \cdots + c_k x_k(t) \]

is also a solution.

- Any linear combination of solutions to the homogeneous system is also a solution.
Linear Independence

**Definition:** A set of $k$ solutions to the linear system $x' = Ax$ is *linearly independent* if they are linearly independent at one value of $t$.

- Proposition $\Rightarrow$ the solutions are linearly independent for all values of $t$. 