Planar Systems

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Procedure to Solve $x' = Ax$

- Find the eigenvalues of $A$, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue $\lambda$ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If $\lambda$ is an eigenvalue and $v$ is an associated nonzero eigenvector, $x(t) = e^{\lambda t}v$ is a solution.
- Show that $n$ of these are linearly independent, if we can.
- This must be explored further.

Planar System $x' = Ax$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

- The characteristic polynomial is $p(\lambda) = \lambda^2 - T\lambda + D$.

where

- $T = \text{tr} A = a_{11} + a_{22}$ and
- $D = \det A = a_{11}a_{22} - a_{12}a_{21}$. 
The eigenvalues of $A$ are the roots of 

$$p(\lambda) = \lambda^2 - T\lambda + D,$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$  

Three cases:

• 2 distinct real roots if $T^2 - 4D > 0$
• 2 complex conjugate roots if $T^2 - 4D < 0$
• Double real root if $T^2 - 4D = 0$

### Real Distinct Eigenvalues

Suppose $A$ is a real $2 \times 2$ matrix with real eigenvalues $\lambda_1 \neq \lambda_2$, and associated nonzero eigenvectors $v_1$ and $v_2$.

Then $x_1(t) = e^{\lambda_1 t}v_1$ and $x_2(t) = e^{\lambda_2 t}v_2$ form a fundamental set of solutions.

### Complex Eigenvalues

Suppose $A$ is a real $2 \times 2$ matrix with complex conjugate eigenvalues $\lambda$ and $\bar{\lambda}$, and associated nonzero eigenvectors $w$ and $\bar{w}$.

Then

• $z(t) = e^{\lambda t}w$ and $\bar{z}(t) = e^{\bar{\lambda} t}\bar{w}$ form a complex valued fundamental set of solutions, and
• $x(t) = \text{Re}(z(t))$ and $y(t) = \text{Im}(z(t))$ form a real valued fundamental set of solutions.
Examples

\[ x' = Ax \]

where

- \[ A = \begin{pmatrix} -4 & 10 \\ -2 & 4 \end{pmatrix} \]
- \[ A = \begin{pmatrix} 7 & 30 \\ -3 & -11 \end{pmatrix} \]

Double Real Root

In this case \( T^2 - 4D = 0 \).

- There is only one eigenvalue
  \[ \lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2} = \frac{T}{2} \]
- The eigenspace of \( \lambda \) has dimension 1 or 2.
  - If the dimension is 2, then \( A = \lambda I \).
  - Every vector is an eigenvector. Every solution has the form
    \[ x(t) = e^{\lambda t}v. \]

Example

\[ x' = Ax \quad \text{where} \quad A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix} \]

- \( p(\lambda) = \lambda^2 + 4\lambda + 4 = (\lambda + 2)^2; \quad \lambda = -2 \)
- \[ A - \lambda I = \begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \quad v = \begin{pmatrix} -3 \\ 1 \end{pmatrix} \]
- The eigenspace has dimension 1, with basis \( v \).
- The standard procedure yields only one solution,
  \[ x_1(t) = e^{\lambda t}v = e^{-2t}(-31)^T. \]
Second Solution

- Look for a second solution of the form
  \[ x_2(t) = e^{\lambda t} [v_2 + tv_1] \]

Then
\[ x'_2 = e^{\lambda t} [(v_1 + \lambda v_2) + \lambda t v_1] \]
\[ A x_2 = e^{\lambda t} [A v_2 + t A v_1] \]

- \( x'_2 = A x_2 \iff A v_1 = \lambda v_1 \) and
  \[ A v_2 = v_1 + \lambda v_2. \]

- We need two things:
  - \( v_1 \) must be an eigenvector.
  - \( (A - \lambda I) v_2 = v_1. \)

The Degenerate Planar Case

- Find the (only) eigenvalue \( \lambda_1. \)
- Find an eigenvector \( v_1 \neq 0. \)
- Find \( v_2 \) with \( (A - \lambda I) v_2 = v_1. \) To do so:
  - Start with any vector \( w \) not a multiple of \( v_1 \)
  - Then \( (A - \lambda I) w = a v_1 \) with \( a \neq 0. \)
  - Set \( v_2 = \frac{1}{a} w. \) \( v_2 \) is not a multiple of \( v_1. \)
- \( x_1(t) = e^{\lambda_1 t} v_1 \) and \( x_2(t) = e^{\lambda_1 t} [v_2 + t v_1] \) form a fundamental set of solutions.

Example (cont.)

- Start with \( w = (1, 0)^T. \)
- \( v_2 = -w = \begin{pmatrix} -1 \\ 0 \end{pmatrix} \)
- Fundamental set of solutions:
  \[ x_1(t) = e^{\lambda_1 t} v_1 = e^{-2t} \begin{pmatrix} -3 \\ 1 \end{pmatrix} \]
  \[ x_2(t) = e^{\lambda_1 t} [v_2 + t v_1] = e^{-2t} \begin{pmatrix} -1 - 3t \\ 0 \end{pmatrix} . \]
Examples

Solve $x' = Ax$, where

- $A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$
- $A = \begin{pmatrix} 0 & 9 \\ -1 & -6 \end{pmatrix}$

Planar System $x' = Ax$

- Equilibrium points for the system
  - Set of equilibrium points equals $\text{null}(A)$.
  - If $A$ nonsingular the only equilibrium point is $0$.
- Can we list the types of all possible equilibrium points for planar linear systems?
  - Six most important cases.
  - Look at solution curves in the phase plane.

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.
  - $\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2}$, $\lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$
- Eigenvectors $v_1$ and $v_2$. The general solution is
  - $x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$
- There are three subcases depending on the signs of the eigenvalues.
Exponential Solutions

\[ x(t) = Ce^{\lambda t}x \]

- The solution curve is a straight half-line through \( Cv \).
- Sometimes called half-line solutions.
- If \( \lambda > 0 \) the solution starts at 0 for \( t = \infty \), and tends to \( \infty \) as \( t \to \infty \). Unstable solution
- If \( \lambda < 0 \) the solution starts at \( \infty \) for \( t = \infty \), and tends to 0 as \( t \to \infty \). Stable solution

Saddle Point

- \( \lambda_1 < 0 < \lambda_2 \)
- General solution \( x(t) = C_1 e^{\lambda_1 t}v_1 + C_2 e^{\lambda_2 t}v_2 \)
- Two stable exponential solutions (\( C_2 = 0 \))
- Two unstable exponential solutions (\( C_1 = 0 \)).
- \( C_1 \neq 0 \) and \( C_2 \neq 0 \).
  - As \( t \to \infty \), \( x(t) \to \infty \), approaching the half-line through \( C_2v_2 \).
  - As \( t \to -\infty \), \( x(t) \to \infty \), approaching the half-line through \( C_2v_1 \).

Nodal Sink

- \( \lambda_1 < \lambda_2 < 0 \)
- General solution \( x(t) = C_1 e^{\lambda_1 t}v_1 + C_2 e^{\lambda_2 t}v_2 \)
- Four stable exponential solutions.
- All solutions \( \to 0 \) as \( t \to \infty \). (Stable)
  - Tangent to \( C_2v_2 \) if \( C_2 \neq 0 \).
  - All solutions \( \to \infty \) as \( t \to -\infty \).
  - \( \parallel \) to the half line through \( C_1v_1 \) if \( C_1 \neq 0 \).
Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $x(t) = C_1 e^{\lambda_1 t}v_1 + C_2 e^{\lambda_2 t}v_2$
- Four unstable exponential solutions.
- All solutions $\to 0$ as $t \to -\infty$.
  - Tangent to $C_1 v_1$ if $C_1 \neq 0$.
  - All solutions $\to \infty$ as $t \to \infty$. (Unstable)
  - $\parallel$ to the half line through $C_2 v_2$ if $C_2 \neq 0$.

Eigenvectors are Linearly Independent

The problem of determining that solutions are linearly independent is eased by the following result.

Proposition: Suppose that $\lambda_1 \neq \lambda_2$ are eigenvalues of the $n \times n$ matrix $A$, and that $v_1 \neq 0$ and $v_2 \neq 0$ are eigenvectors associated with $\lambda_1$ and $\lambda_2$, respectively. Then $v_1$ and $v_2$ are linearly independent.