Math 211

Lecture #28

Phase Plane Portraits

November 1, 2002
Procedure to Solve $x' = Ax$

- Find the eigenvalues of $A$, which are the roots of $\det(A - \lambda I) = 0$.
- For each eigenvalue $\lambda$ find the eigenspace, which is equal to $\text{null}(A - \lambda I)$.
- If $\lambda$ is an eigenvalue and $v$ is an associated nonzero eigenvector, $x(t) = e^{\lambda t}v$ is a solution.
- Show that $n$ of these are linearly independent, if we can.
  - This must be explored further if the system has dimension $n \geq 3$. 
Planar System $x' = Ax$

$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$

- The characteristic polynomial is 

$$p(\lambda) = \lambda^2 - T\lambda + D.$$ 

where

- $T = \text{tr} A = a_{11} + a_{22}$ and
- $D = \det A = a_{11}a_{22} - a_{12}a_{21}$. 
• The eigenvalues of $A$ are the roots of

$$p(\lambda) = \lambda^2 - T\lambda + D,$$

$$\lambda = \frac{T \pm \sqrt{T^2 - 4D}}{2}.$$ 

• Three cases:

♦ 2 distinct real roots if $T^2 - 4D > 0$

♦ 2 complex conjugate roots if $T^2 - 4D < 0$

♦ Double real root if $T^2 - 4D = 0$
Real Distinct Eigenvalues

Suppose $A$ is a real $2 \times 2$ matrix with real eigenvalues $\lambda_1 \neq \lambda_2$, and associated nonzero eigenvectors $v_1$ and $v_2$.

Then $x_1(t) = e^{\lambda_1 t}v_1$ and $x_2(t) = e^{\lambda_1 t}v_2$ form a fundamental set of solutions.
Complex Eigenvalues

Suppose $A$ is a real $2 \times 2$ matrix with complex conjugate eigenvalues $\lambda$ and $\bar{\lambda}$, and associated nonzero eigenvectors $w$ and $\bar{w}$.

Then

- $z(t) = e^{\lambda t}w$ and $\bar{z}(t) = e^{\bar{\lambda} t}\bar{w}$ form a complex valued fundamental set of solutions, and

- $x(t) = \text{Re}(z(t))$ and $y(t) = \text{Im}(z(t))$ form a real valued fundamental set of solutions.
The Degenerate Planar Case

- Find the (only) eigenvalue $\lambda_1$.
- Find an eigenvector $v_1 \neq 0$.
- Find $v_2$ with $(A - \lambda I)v_2 = v_1$. To do so:
  - Start with any vector $w$ not a multiple of $v_1$
  - Then $(A - \lambda I)w = av_1$ with $a \neq 0$.
  - Set $v_2 = \frac{1}{a}w$. $v_2$ is not a multiple of $v_1$.
- $x_1(t) = e^{\lambda t}v_1$ and $x_2(t) = e^{\lambda t}[v_2 + tv_1]$ form a fundamental set of solutions.
Qualitative Analysis for a Planar System

$$x' = Ax$$

- Equilibrium points for the system
  - The set of equilibrium points equals $\text{null}(A)$.
  - If $A$ is nonsingular, the only equilibrium point is $0$.

- Can we list the types of all possible equilibrium points for planar linear systems?
  - The cases are distinguished by different solution curves in the phase plane.
  - We will do the six most important cases.
Distinct Real Eigenvalues

- \( p(\lambda) = \lambda^2 - T\lambda + D \) with \( T^2 - 4D > 0 \).

\[
\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}
\]

- Eigenvectors \( \mathbf{v}_1 \) and \( \mathbf{v}_2 \). The general solution is

\[
\mathbf{x}(t) = C_1 e^{\lambda_1 t} \mathbf{v}_1 + C_2 e^{\lambda_2 t} \mathbf{v}_2
\]

- There are three subcases depending on the signs of the eigenvalues.
Exponential Solutions

\[ x(t) = Ce^{\lambda t}v \]

- The solution curve is a straight half-line through \( Cv \). Sometimes called \textit{half-line} solutions.
- If \( \lambda > 0 \) the solution starts at 0 for \( t = -\infty \), and tends to \( \infty \) as \( t \to \infty \). This is an \textit{unstable solution}
- If \( \lambda < 0 \) the solution starts at \( \infty \) for \( t = -\infty \), and tends to 0 as \( t \to \infty \). This is a \textit{stable solution}
Saddle Point

• $\lambda_1 < 0 < \lambda_2$
• General solution $x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$
• Two stable exponential solutions ($C_2 = 0$)
• Two unstable exponential solutions ($C_1 = 0$).
• Suppose that $C_1 \neq 0$ and $C_2 \neq 0$.
  ♦ As $t \to \infty$, $x(t) \to \infty$, approaching the half-line through $C_2 v_2$.
  ♦ As $t \to -\infty$, $x(t) \to \infty$, approaching the half-line through $C_2 v_1$. 
Nodal Sink

- $\lambda_1 < \lambda_2 < 0$
- General solution $x(t) = C_1 e^{\lambda_1 t}v_1 + C_2 e^{\lambda_2 t}v_2$
- Four stable exponential solutions.
- All solutions $\to 0$ as $t \to \infty$. (Stable)
  - Tangent to $C_2v_2$ if $C_2 \neq 0$.
- All solutions $\to \infty$ as $t \to -\infty$.
  - || to the half line through $C_1v_1$ if $C_1 \neq 0$. 
Nodal Source

- $0 < \lambda_1 < \lambda_2$
- General solution $x(t) = C_1 e^{\lambda_1 t} v_1 + C_2 e^{\lambda_2 t} v_2$
- Four unstable exponential solutions.
- All solutions $\to 0$ as $t \to -\infty$.
  ♦ Tangent to $C_1 v_1$ if $C_1 \neq 0$.
- All solutions $\to \infty$ as $t \to \infty$. (Unstable)
  ♦ $\parallel$ to the half line through $C_2 v_2$ if $C_2 \neq 0$. 
Complex Eigenvalues

- \( p(\lambda) = \lambda^2 - T\lambda + D \) with \( T^2 - 4D < 0 \)
  
  \[ \lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta. \]

- Eigenvector \( \mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2 \) associated to \( \lambda \).

- Complex solutions
  
  \[ \mathbf{z}(t) = e^{\lambda t} \mathbf{w} = e^{t(\alpha+i\beta)}[\mathbf{v}_1 + i\mathbf{v}_2] \]
  
  \[ \overline{\mathbf{z}}(t) = e^{\overline{\lambda} t} \overline{\mathbf{w}} = e^{t(\alpha-i\beta)}[\mathbf{v}_1 - i\mathbf{v}_2] \]
• **Real solutions**

\[
x_1(t) = \text{Re}(z(t)) = e^{\alpha t} [\cos \beta t \cdot v_1 - \sin \beta t \cdot v_2]
\]

\[
x_2(t) = \text{Im}(z(t)) = e^{\alpha t} [\sin \beta t \cdot v_1 + \cos \beta t \cdot v_2]
\]

• **General solution**

\[
x(t) = C_1 e^{\alpha t} [\cos \beta t \cdot v_1 - \sin \beta t \cdot v_2]
\]

\[
+ C_2 e^{\alpha t} [\sin \beta t \cdot v_1 + \cos \beta t \cdot v_2]
\]

• There are three cases depending on the sign of \(\alpha = \text{Re}(\lambda)\).
Center

- $\alpha = \text{Re}(\lambda) = 0$

- The general real solution is

$$\mathbf{x}(t) = C_1 [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2]$$

$$+ C_2 [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- Every solution is periodic with period $T = \frac{2\pi}{\beta}$.

- All solution curves are ellipses.
Spiral Sink

- $\alpha = \text{Re}(\lambda) < 0$

- The general real solution is

$$x(t) = C_1 e^{\alpha t} [\cos \beta t \cdot v_1 - \sin \beta t \cdot v_2] + C_2 e^{\alpha t} [\sin \beta t \cdot v_1 + \cos \beta t \cdot v_2]$$

- All solutions spiral into 0 as $t \to \infty$. 
Spiral Source

- $\alpha = \text{Re}(\lambda) > 0$

- The general real solution is

$$\mathbf{x}(t) = C_1 e^{\alpha t} [\cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2] + C_2 e^{\alpha t} [\sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2]$$

- All solutions spiral into 0 as $t \to -\infty$. 
Planar Systems

\[ A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

- The characteristic polynomial is

\[ p(\lambda) = \lambda^2 - T\lambda + D. \]

where

- \( T = \text{tr} \, A = a_{11} + a_{22} \) and

- \( D = \det \, A = a_{11}a_{22} - a_{12}a_{21} \).

- The eigenvalues are

\[ \lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2}. \]
• \(\lambda_1 \& \lambda_2\) are the roots of \(p(\lambda) = \lambda^2 - T\lambda + D\), so

\[
p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \]

\[
= \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2
\]

• Hence, \(T = \lambda_1 + \lambda_2\) and \(D = \lambda_1\lambda_2\).

• Duality between \((\lambda_1, \lambda_2)\) and \((T, D)\).

• We will represent a system by the location of \((T, D)\) in the \(TD\)-plane — the trace-determinant plane.
Trace-Determinant Plane

- $T^2 - 4D > 0$
  - $\Rightarrow$ distinct real eigenvalues $\lambda_1$ & $\lambda_2$

- $D = \lambda_1 \lambda_2 < 0 \Rightarrow$ Saddle point.

- $D = \lambda_1 \lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
  - $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
  - $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.
\[ T^2 - 4D < 0 \Rightarrow \text{complex eigenvalues} \]

\[ \lambda = \alpha + i\beta \quad \text{and} \quad \overline{\lambda} = \alpha - i\beta. \]

\[ T = \lambda + \overline{\lambda} = 2\alpha > 0 \Rightarrow \text{Spiral source.} \]

\[ T = \lambda + \overline{\lambda} = 2\alpha < 0 \Rightarrow \text{Spiral sink.} \]

\[ T = \lambda + \overline{\lambda} = 2\alpha = 0 \Rightarrow \text{Center.} \]
Types of Equilibrium Points

- **Generic** types
  - Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.

- **Nongeneric** types
  - Center and many others. Occupy pieces of the boundaries between the generic types.