Planar System $\mathbf{x}' = A\mathbf{x}$

- Equilibrium points for the system
- Set of equilibrium points equals $\text{null}(A)$.
- $A$ nonsingular $\Rightarrow$ only equilibrium point is $0$.
- Can we list the types of all possible equilibrium points for planar linear systems?
  - We will do the six most important cases.
  - Look at solution curves in the phase plane.

Distinct Real Eigenvalues

- $p(\lambda) = \lambda^2 - T\lambda + D$ with $T^2 - 4D > 0$.
  $$\lambda_1 = \frac{T - \sqrt{T^2 - 4D}}{2} < \lambda_2 = \frac{T + \sqrt{T^2 - 4D}}{2}$$
- Eigenvectors $\mathbf{v}_1$ and $\mathbf{v}_2$. General solution
  $$\mathbf{x}(t) = C_1 e^{\lambda_1 t}\mathbf{v}_1 + C_2 e^{\lambda_2 t}\mathbf{v}_2$$
  - $\lambda_1 < 0 < \lambda_2$ Saddle point.
  - $\lambda_1 < \lambda_2 < 0$ Nodal sink.
  - $0 < \lambda_1 < \lambda_2$ Nodal source.
Complex Eigenvalues

- \( p(\lambda) = \lambda^2 - T\lambda + D \) with \( T^2 - 4D < 0 \)
  \[ \lambda = \alpha + i\beta \quad \text{and} \quad \bar{\lambda} = \alpha - i\beta. \]

- Eigenvector \( \mathbf{w} = \mathbf{v}_1 + i\mathbf{v}_2 \) associated to \( \lambda \).
- General solution
  \[ \mathbf{x}(t) = C_1 e^{\alpha t} \left[ \cos \beta t \cdot \mathbf{v}_1 - \sin \beta t \cdot \mathbf{v}_2 \right] 
  + C_2 e^{\alpha t} \left[ \sin \beta t \cdot \mathbf{v}_1 + \cos \beta t \cdot \mathbf{v}_2 \right] \]

- \( \alpha = \text{Re}(\lambda) = 0 \) Center.
- \( \alpha = \text{Re}(\lambda) < 0 \) Spiral sink.
- \( \alpha = \text{Re}(\lambda) > 0 \) Spiral source.

Planar Systems

\[ \mathbf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \]

- The characteristic polynomial is \( p(\lambda) = \lambda^2 - T\lambda + D \).
  where
  - \( T = \text{tr} \mathbf{A} = a_{11} + a_{22} \) and
  - \( D = \det \mathbf{A} = a_{11}a_{22} - a_{12}a_{21} \).
- The eigenvalues are
  \[ \lambda_1, \lambda_2 = \frac{T \pm \sqrt{T^2 - 4D}}{2} \]

- \( \lambda_1 \) and \( \lambda_2 \) are the roots of \( p(\lambda) = \lambda^2 - T\lambda + D \), so
  \[ p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 \]
- Hence, \( T = \lambda_1 + \lambda_2 \) and \( D = \lambda_1\lambda_2 \).
- Duality between \( \lambda_1, \lambda_2 \) and \( T, D \).
- We will represent a system by the location of \( (T, D) \) in the \( TD \)-plane — the trace-determinant plane.
Trace-Determinant Plane

- $T^2 - 4D > 0$
  - $\Rightarrow$ distinct real eigenvalues $\lambda_1$ and $\lambda_2$
  - $D = \lambda_1\lambda_2 < 0 \Rightarrow$ Saddle point.
  - $D = \lambda_1\lambda_2 > 0 \Rightarrow$ Eigenvalues have the same sign.
    - $T = \lambda_1 + \lambda_2 > 0 \Rightarrow$ Nodal source.
    - $T = \lambda_1 + \lambda_2 < 0 \Rightarrow$ Nodal sink.

- $T^2 - 4D < 0 \Rightarrow$ complex eigenvalues
  - $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$.
    - $T = \lambda + \bar{\lambda} = 2\alpha > 0 \Rightarrow$ Spiral source.
    - $T = \lambda + \bar{\lambda} = 2\alpha < 0 \Rightarrow$ Spiral sink.
    - $T = \lambda + \bar{\lambda} = 2\alpha = 0 \Rightarrow$ Center.

Types of Equilibrium Points

- Generic types
  - Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.

- Nongeneric types
  - Center and many others. Occupy pieces of the boundaries between the generic types.
Higher Dimensional Systems

\[ x' = Ax \]

- \( A \) is a real \( n \times n \) matrix.
- If \( \lambda \) is an eigenvalue and \( v \not= 0 \) is an associated eigenvector, then
  \[ x(t) = e^{\lambda t}v \]
  is a solution.
- Much like the planar case, but now we need \( n \) linearly independent solutions.
- We no longer have the easy way to compute the characteristic polynomial
  \( p(\lambda) = \det(A - \lambda I) \).

Proposition: Suppose that \( \lambda_1, \ldots, \lambda_k \) are distinct eigenvalues of \( A \), and that \( v_1, \ldots, v_k \) are associated nonzero eigenvectors. Then \( v_1, \ldots, v_k \) are linearly independent.

Theorem: Suppose the \( n \times n \) real matrix \( A \) has \( n \) distinct eigenvalues \( \lambda_1, \ldots, \lambda_n \), and that \( v_1, \ldots, v_n \) are associated nonzero eigenvectors. Then the exponential solutions
\[ x_i(t) = e^{\lambda_i t}v_i, \quad 1 \leq i \leq n \]
form a fundamental set of solutions for the system \( x' = Ax \).

Examples:
- \( A = \begin{pmatrix} -2 & 3 & -4 \\ 0 & 1 & 0 \\ 0 & 4 & -1 \end{pmatrix} \)
- \( A = \begin{pmatrix} 17 & -30 & -8 \\ 16 & -29 & -8 \\ -12 & 24 & 7 \end{pmatrix} \)
- Use MATLAB.
Complex Eigenvalues

A real $n \times n$ matrix with a complex eigenvalue $\lambda$ and associate eigenvector $w$.

- $\Rightarrow \overline{\lambda}$ is an eigenvalue and $\overline{w}$ is an associated nonzero eigenvector.

- Complex valued solutions: $x(t) = e^{\lambda t}w$
  $\overline{x}(t) = e^{\overline{\lambda} t}\overline{w}$.

- Real solutions: $x(t) = \text{Re}(x(t))$
  $y(t) = \text{Im}(x(t))$.

Example

$$A = \begin{pmatrix} 21 & 10 & 4 \\ -70 & -31 & -10 \\ 30 & 10 & -1 \end{pmatrix}$$

- The theorem applies if some of the eigenvalues are complex and we replace complex conjugate pairs of solutions by their real and imaginary parts.

Repeated Eigenvalues – Example 1

$$A = \begin{pmatrix} -5 & -10 & 6 \\ 8 & 19 & -12 \\ 12 & 30 & -19 \end{pmatrix}$$

- $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
- $\lambda_1 = -3$
  - Eigenspace has dimension 1 $\Rightarrow$ one exponential solution
  $$x_1(t) = e^{-3t}(-1/3, 2/3, 1)^T$$
• $\lambda_2 = -1$
  • Eigenspace has dimension 2 $\Rightarrow$ two linearly independent exponential solutions
  • Eigenspace has basis $v_2 = (-5/2, 1, 0)^T$ and $v_3 = (3/2, 0, 1)^T$.
  • Linearly independent solutions
    \[
    x_2(t) = e^{-t} \begin{pmatrix} -5/2 \\ 1 \\ 0 \end{pmatrix} \quad \& \quad x_3(t) = e^{-t} \begin{pmatrix} 3/2 \\ 0 \\ 1 \end{pmatrix}
    \]
  • $x_1$, $x_2$, and $x_3$ are a fundamental set of solutions.

Repeated Eigenvalues – Example 2

\[
A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix}
\]

• $p(\lambda) = (\lambda + 3)(\lambda + 1)^2$
  • $\lambda_1 = -3$
  • Eigenspace has dimension 1 $\Rightarrow$ one exponential solution
    \[
    x_1(t) = e^{-3t}(-1/2, 3/2, 1)^T
    \]

• $\lambda_2 = -1$
  • Eigenspace has dimension 1 $\Rightarrow$ only one exponential solution
    \[
    x_2(t) = e^{-t} \begin{pmatrix} -1/2 \\ 1 \\ 1 \end{pmatrix}
    \]
  • Need a third solution.
  • Need a new idea.
Multiplicities

A $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial is
  
  $$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}.$$  

- The algebraic multiplicity of $\lambda_j$ is $q_j$.
- The geometric multiplicity of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$.

We always have:

- $q_1 + q_2 + \cdots + q_k = n$.
- $1 \leq d_j \leq q_j$.
- There are $d_j$ linearly independent exponential solutions corresponding to $\lambda_j$.
- If $d_j = q_j$ for all $j$ we have $n$ linearly independent solutions.
- If $d_j < q_j$ we have trouble.