Math 211

Lecture #30

The Exponential of a Matrix

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Repeated Eigenvalues – Example 1

\[
A = \begin{pmatrix}
-5 & -10 & 6 \\
8 & 19 & -12 \\
12 & 30 & -19
\end{pmatrix}
\]

- \( p(\lambda) = (\lambda + 3)(\lambda + 1)^2 \)
- \( \lambda_1 = -3 \) : Eigenspace has dimension 1, with basis \( v_1 \), so there is one exponential solution, \( x_1(t) = e^{\lambda_1 t}v_1 \).
- \( \lambda_2 = -1 \) : Eigenspace has dimension 2 with basis \( v_2 \) and \( v_3 \), so there are two linearly independent exponential solutions \( x_2(t) = e^{\lambda_2 t}v_2 \) and \( x_3(t) = e^{\lambda_2 t}v_3 \).
- \( x_1, x_2, \) and \( x_3 \) are a fundamental set of solutions.
Repeated Eigenvalues – Example 2

\[ A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix} \]

- \[ p(\lambda) = (\lambda + 3)(\lambda + 1)^2 \]
- \( \lambda_1 = -3 \): Eigenspace has dimension 1. There is one exponential solution \( x_1(t) = e^{-3t}(-1/2, 3/2, 1)^T \).
- \( \lambda_2 = -1 \): Eigenspace has dimension 1. There is only one exponential solution \( x_2(t) = e^{-t}(-1/2, 1, 1)^T \).
- We need a third solution. We need a new idea.
Multiplicities

$A$ an $n \times n$ matrix

- Distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.
- The characteristic polynomial factors as
  \[ p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}. \]
- The \textit{algebraic multiplicity} of $\lambda_j$ is $q_j$.
- The \textit{geometric multiplicity} of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$. 
• We always have:
  
  • \( q_1 + q_2 + \cdots + q_k = n. \)
  
  • \( 1 \leq d_j \leq q_j. \)
  
  • There are \( d_j \) linearly independent exponential solutions corresponding to \( \lambda_j. \)
  
  • If \( d_j = q_j \) for all \( j \) we have \( n \) linearly independent solutions.
  
  • If \( d_j < q_j \) we have trouble.
New Approach

- \( D = 1 : x' = ax \)
  - Solution \( x(t) = Ce^{at}. \)

- \( D > 1 : x' = Ax \)
  - Tried \( x(t) = e^{\lambda t}v. \)
    - Worked well except when eigenvalues have multiplicity greater than 1.
  - Why not \( x(t) = e^{tA}v? \)

- But what is \( e^{tA}? \)
Exponential of a Matrix

**Definition:** The *exponential* of the $n \times n$ matrix $A$ is the $n \times n$ matrix

\[
e^A = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots
\]

\[
= \sum_{0}^{\infty} \frac{1}{n!} A^n.
\]

**Examples:**

- $A = \begin{pmatrix} r_1 & 0 \\ 0 & r_2 \end{pmatrix} \Rightarrow e^A = \begin{pmatrix} e^{r_1} & 0 \\ 0 & e^{r_2} \end{pmatrix}$.
- $e^{\lambda I} = e^{\lambda I}$.  $e^{0 I} = I$. 
Properties

- $A$ commutes with $e^A$,

$$Ae^A = e^A A.$$

- If $A$ and $B$ commute (i.e., $AB = BA$), then

$$e^{A+B} = e^A \cdot e^B.$$

- The inverse of $e^A$ is $e^{-A}$.

- $\frac{d}{dt} e^{tA} = Ae^{tA}$. 
A Very Important Fact

**Theorem:** The solution to the initial value problem

\[ x' = Ax \text{ with } x(0) = v \]

is given by \( x(t) = e^{tA}v \).

- However computing \( e^{tA} \) is not easy.
Suppose that $A$ an $n \times n$ matrix, and $\lambda$ a number (an eigenvalue).

- Then $A = \lambda I + (A - \lambda I)$, and $\lambda I$ & $A - \lambda I$ commute. Therefore

  $$e^{tA} = e^{t[\lambda I+(A-\lambda I)]}$$

  $$= e^{t\lambda I} \cdot e^{t(A-\lambda I)}$$

  $$= e^{\lambda t} \cdot e^{t(A-\lambda I)}$$

  $$= e^{\lambda t} \cdot [I + t(A - \lambda I) + \frac{t^2}{2!} (A - \lambda I)^2 + \cdots]$$
\[ e^{tA}v, \; v \text{ an Eigenvector} \]

Let \( \lambda \) be an eigenvalue and \( v \) an associated eigenvector. Then \((A - \lambda I)v = 0\), so

\[ e^{tA}v = e^{\lambda t} \cdot e^{t(A-\lambda I)}v \]

\[ = e^{\lambda t}[I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots]v \]

\[ = e^{\lambda t}[v + t(A - \lambda I)v + \frac{t^2}{2!}(A - \lambda I)^2v + \cdots] \]

\[ = e^{\lambda t}v \]

- The infinite series truncates, so we can compute \( e^{tA}v \).
Matrices with One Eigenvalue

A has characteristic polynomial \( p(\lambda) = (\lambda - \lambda_1)^n \).

- **Cayley-Hamilton Theorem:** If \( p(\lambda) \) is the characteristic polynomial of the matrix \( A \) then \( p(A) = 0I \).

- In our case \((A - \lambda_1 I)^n = 0I\), so

\[
e^{tA} = e^{\lambda_1 t} \cdot [I + t(A - \lambda_1 I) + \frac{t^2}{2!}(A - \lambda_1 I)^2 + \cdots + \frac{t^{n-1}}{(n-1)!}(A - \lambda_1 I)^{n-1}]
\]
Example 3

\[ A = \begin{pmatrix} -3 & 1 \\ -1 & -1 \end{pmatrix} \]

- \[ p(\lambda) = (\lambda + 2)^2. \]

\[ A + 2I = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}, \quad (A + 2I)^2 = 0I \]

\[ e^{tA} = e^{-2t}[I + t(A + 2I)] \]

\[ = e^{-2t} \begin{pmatrix} 1 - t & t \\ -t & 1 + t \end{pmatrix}. \]
Example 4

\[ A = \begin{pmatrix} 0 & -9 & 27 \\ -2 & 3 & -18 \\ -1 & 3 & -12 \end{pmatrix} \]

- \( p(\lambda) = (\lambda + 3)^3 \). \( (A + 3I)^2 = 0I \).

\[
e^{tA} = e^{-3t}[I + t(A + 3I)]
\]

\[
= e^{-3t} \begin{pmatrix} 1 + 3t & -9t & 27t \\ -2t & 1 + 6t & -18t \\ -t & 3t & 1 - 9t \end{pmatrix}.
\]
Example 2, Reprise

• Distinct eigenvalues $\lambda_1 = -3 \& \lambda_2 = -1$

• Different from previous two examples.

• $\lambda_1 = -3$ has algebraic multiplicity 1, and geometric multiplicity 1. So there is one exponential solution

$$x_1(t) = e^{\lambda_1 t}v_1 = e^{-3t}(-1/2, 3/2, 1)^T.$$

• $\lambda_2 = -1$ has algebraic multiplicity 2, and geometric multiplicity 1. So there is only one exponential solution

$$x_2(t) = e^{\lambda_2 t}v_2 = e^{-t}(-1/2, 1, 1)^T.$$
• However, \( \text{null}((A - \lambda_2 I)^2) \) has dimension 2, with basis \((0, 1, 1)^T\) and \((1, 0, 0)^T\). If \( \mathbf{v} \in \text{null}((A - \lambda_2 I)^2) \) then

\[
e^{tA}\mathbf{v} = e^{\lambda_2 t}[I + t(A - \lambda_2 I) + \frac{t^2}{2!}(A - \lambda_2 I)^2 + \cdots] \mathbf{v}
\]

\[
= e^{\lambda_2 t}[\mathbf{v} + t(A - \lambda_2 I)\mathbf{v}].
\]

• \( \mathbf{v}_2 \) is in \( \text{null}((A - \lambda_2 I)^2) \)

• Using \( \mathbf{v}_3 = (1, 0, 0)^T \) we get the third solution

\[
x_3(t) = e^{tA}\mathbf{v}_3 = e^{-t}[\mathbf{v}_3 + t(A + I)\mathbf{v}_3]
\]

\[
= e^{-t}(1 + 2t, -4t, -4t)^T.
\]

• \( x_1, x_2, \) and \( x_3 \) are a fundamental set of solutions.
Summary

• In Examples 3 & 4 the matrix has one eigenvalue.
  ♦ The series for $e^{t(A-\lambda I)}$ truncates to a finite sum.

• In Example 2 the matrix had two eigenvalues.
  ♦ The series for $e^{t(A-\lambda I)}$ does not truncate for any $\lambda$.
  ♦ However, the series for $e^{t(A-\lambda_2 I)}\mathbf{v}$ does truncate if $(A - \lambda_2 I)^2\mathbf{v} = 0$. 
Generalized Eigenvectors

**Definition:** If \( \lambda \) is an eigenvalue of \( A \) and
\[
(A - \lambda I)^p v = 0
\]
for some integer \( p \geq 1 \), then \( v \) is called a generalized eigenvector associated with \( \lambda \).

- The series for \( e^{t(A - \lambda I)} v \) truncates to a finite sum if \( v \) is a generalized eigenvector associated with \( \lambda \).
- We can compute \( e^{tA} v \).
**Theorem:** If $\lambda$ is an eigenvalue of $A$ with algebraic multiplicity $q$, then there is an integer $p \leq q$ such that $\text{null}((A - \lambda I)^p)$ has dimension $q$.

- For each generalized eigenvector $v$ we can compute $e^{tA}v$.
- We can find $q$ linearly independent solutions associated with the eigenvalue $\lambda$. 
Procedure for $\lambda$ of algebraic multiplicity $q$

To find $q$ linearly independent solutions associated with $\lambda$:

- Find the smallest integer $p$ such that $\text{null}((A - \lambda I)^p)$ has dimension $q$.
- Find a basis $v_1, v_2, \ldots, v_q$ of $\text{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$

\[ x_j(t) = e^{tA}v_j \]

\[ = e^{\lambda t}[v_j + t(A - \lambda I)v_j + \frac{t^2}{2!}(A - \lambda I)^2v_j \]

\[ + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}v_j] \]
Example

- Use MATLAB.
Procedure for a Complex Eigenvalue

If $\lambda$ is complex of algebraic multiplicity $q$. Then $\bar{\lambda}$ also has multiplicity $q$.

- Find the smallest integer $p$ such that $\text{null}((A - \lambda I)^p)$ has dimension $q$.
- Find a basis $w_1, w_2, \ldots, w_q$ of $\text{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$
  \[ z_j(t) = e^{tA}w. \]
For $j = 1, 2, \ldots, q$

\[
z_j(t) = e^{\lambda t} \left[ w_j + t(A - \lambda I)w_j + \frac{t^2}{2!} (A - \lambda I)^2 w_j + \cdots + \frac{t^{p-1}}{(p-1)!} (A - \lambda I)^{p-1} w_j \right]
\]

For $j = 1, 2, \ldots, q$ set

\[
x_j(t) = \text{Re}(z_j(t)) \quad \text{and} \quad y_j(t) = \text{Im}(z_j(t)).
\]