Math 211

Lecture #31
Exponential of a Matrix
Stability of Solutions

November 8, 2002

Exponential of a Matrix

Definition: The exponential of the $n \times n$ matrix $A$ is the $n \times n$ matrix

$$e^A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}A^n.$$  

Theorem: The solution to the initial value problem

$$x' = Ax \quad \text{with} \quad x(0) = v$$

is $x(t) = e^{tA}v$.

- Can we compute $e^{tA}v$ for enough vectors to find a fundamental set of solutions?

Key to Computing $e^{tA}$ or $e^{tA}v$

Suppose that $A$ an $n \times n$ matrix, and $\lambda$ a number (an eigenvalue). Then

$$e^{tA} = e^{t\lambda} \cdot [I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots]$$

$$e^{tA}v = e^{t\lambda} \cdot [v + t(A - \lambda I)v + \frac{t^2}{2!}(A - \lambda I)^2v + \cdots]$$

- If $\lambda$ is an eigenvalue and $v$ is an associated eigenvector, then $e^{tA}v = e^{t\lambda}v$.
- If $(A - \lambda I)^2v = 0$, then $e^{tA}v = e^{t\lambda}v + t(A - \lambda I)v$.
Example 2, Reprise

\[ A = \begin{pmatrix} 1 & 2 & -1 \\ -4 & -7 & 4 \\ -4 & -4 & 1 \end{pmatrix} \]

- \( p(\lambda) = (\lambda + 3)(\lambda + 1)^2 \)
- \( \lambda_1 = -3 \), with algebraic multiplicity 1.
  - \( \text{null}(A - \lambda_1 I) \) has basis \( v_1 = (-1/2, 3/2, 1)^T \), so the geometric multiplicity is 1.
  - There is one exponential solution
    \[ x_1(t) = e^{\lambda_1}v_1 = e^{-3}(-1/2, 3/2, 1)^T. \]

- \( \lambda_2 = -1 \), with algebraic multiplicity 2.
  - \( \text{null}(A - \lambda_2 I) \) has basis \( v_2 = (-1/2, 1, 1)^T \), so the geometric multiplicity is 1.
  - So there is only one exponential solution
    \[ x_2(t) = e^{\lambda_2}v_2 = e^{-t}(-1/2, 1, 1)^T. \]
  - However, \( \text{null}((A - \lambda_2 I)^2) \) has dimension 2, with basis \((0, 1, 1)^T\) and \((1, 0, 0)^T\). With \( v_3 = (1, 0, 0)^T \) we get the third solution
    \[ x_3(t) = e^{\lambda_3}v_3 = e^{-t}[v_3 + t(A + I)v_3] \]
    \[ = e^{-t}(1 + 2t, -4t, -4t)^T. \]
- \( x_1, x_2, \) and \( x_3 \) are a fundamental set of solutions.

Generalized Eigenvectors

Definition: If \( \lambda \) is an eigenvalue of \( A \) and \((A - \lambda I)^p v = 0 \) for some integer \( p \geq 1 \), then \( v \) is called a generalized eigenvector associated with \( \lambda \).

Then
\[ e^{tA}v = e^{\lambda t} \left[ v + t(A - \lambda I)v + \frac{t^2}{2!}(A - \lambda I)^2v \right. \]
\[ + \cdots + \frac{t^{p-1}}{(p-1)!}(A - \lambda I)^{p-1}v \]

- We can compute \( e^{tA}v \) for any generalized eigenvector.
Solution Strategy

Theorem: If \( \lambda \) is an eigenvalue of \( A \) with algebraic multiplicity \( q \), then there is an integer \( p \leq q \) such that \( \text{null}((A - \lambda I)^p) \) has dimension \( q \).

- Thus, we can find \( q \) linearly independent solutions associated with the eigenvalue \( \lambda \).
- Since the sum of the algebraic multiplicities is \( n \), we can find a fundamental set of solutions.

Procedure for Solving \( x' = Ax \)

- Find the eigenvalues of \( A \).
- For each eigenvalue \( \lambda \):
  - Find the algebraic multiplicity \( q \).
  - Find the smallest integer \( p \) such that \( \text{null}((A - \lambda I)^p) \) has dimension \( q \).
  - Find a basis \( v_1, v_2, \ldots, v_q \) of \( \text{null}((A - \lambda I)^p) \).
  - For \( j = 1, 2, \ldots, q \), set \( x_j(t) = e^{tA}v_j \).
  - If \( \lambda \) is complex, find real solutions.

Examples

- Use MATLAB.
Procedure for a Complex Eigenvalue

If $\lambda$ is a complex eigenvalue of algebraic multiplicity $q$.
Then $\lambda$ also has algebraic multiplicity $q$.

- Find the smallest integer $p$ such that $\text{null}((A - \lambda I)^p)$ has dimension $q$.
- Find a basis $w_1, w_2, \ldots, w_q$ of $\text{null}((A - \lambda I)^p)$.
- For $j = 1, 2, \ldots, q$, set $z_j(t) = e^{t \lambda} w_j$, $z_1, \ldots, z_q$.
Together with $z_j$, $z_1, \ldots, z_q$, these are $2q$ linearly independent complex valued solutions.
- For $j = 1, 2, \ldots, q$, set $x_j(t) = \text{Re}(z_j(t))$ and $y_j(t) = \text{Im}(z_j(t))$. These are $2q$ linearly independent real valued solutions.

Stability

Autonomous system $x' = f(x)$ with an equilibrium point at $x_0$.

- Basic question: What happens to all solutions as $t \to \infty$?
- $x_0$ is stable if for every $\epsilon > 0$ there is a $\delta > 0$ such that a solution $x(t)$ with $|x(0) - x_0| < \delta \Rightarrow |x(t) - x_0| < \epsilon$ for all $t \geq 0$.
- Every solution that starts close to $x_0$ stays close to $x_0$.

- $x_0$ is asymptotically stable if it is stable and there is an $\eta > 0$ such that if $x(t)$ is a solution with $|x(0) - x_0| < \eta$, then $x(t) \to x_0$ as $t \to \infty$.
- $x_0$ is called a sink.
- Every solution that starts close to $x_0$ approaches $x_0$.
- $x_0$ is unstable if there is an $\epsilon > 0$ such that for any $\delta > 0$ there is a solution $x(t)$ with $|x(0) - x_0| < \delta$ with the property that there are values of $t > 0$ such that $|x(t) - x_0| > \epsilon$.
- There are solutions starting arbitrarily close to $x_0$ that move away from $x_0$. 
**Examples $D = 2$**

- Sinks are asymptotically stable.
  - The eigenvalues have negative real part.
- Sources are unstable.
  - The eigenvalues have positive real part.
- Saddles are unstable.
  - One eigenvalue has positive real part.
- Centers are stable but not asymptotically stable.
  - The eigenvalues have real part $= 0$.

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**Theorem:** Let $A$ be an $n \times n$ real matrix.

- Suppose the real part of every eigenvalue of $A$ is negative. Then $0$ is an asymptotically stable equilibrium point for the system $x' = Ax$.
- Suppose $A$ has at least one eigenvalue with positive real part. Then $0$ is an unstable equilibrium point for the system $x' = Ax$.

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**Examples**

- $D = 2$
  - $T^2 - 4D = 0$.
    - $T < 0 \Rightarrow$ sink. $T > 0 \Rightarrow$ source.
  - $y' = Ay$,
    $$A = \begin{pmatrix} -2 & -18 & -7 & -14 \\ 1 & 6 & 2 & 5 \\ 2 & 2 & -3 & 0 \\ -2 & -8 & -1 & -6 \end{pmatrix}.$$
    - $A$ has eigenvalues $-1$, $-2$, & $-1 \pm i$.
    - $0$ is asymptotically stable.
Multiplicities

An $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.

- The characteristic polynomial has the form
  $$p(\lambda) = (\lambda - \lambda_1)^{q_1}(\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}.$$  

- The algebraic multiplicity of $\lambda_j$ is $q_j$.
- The geometric multiplicity of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$.
- $q_1 + q_2 + \ldots + q_k = n$.
- $1 \leq d_j \leq q_j$. 

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