Inhomogeneous Equations

Theorem: Assume

• $y_p(t)$ is a particular solution to the inhomogeneous equation $y'' + py' + qy = f(t)$;
• $y_1(t)$ & $y_2(t)$ is a fundamental set of solutions to the homogeneous equation $y'' + py' + qy = 0$.

Then the general solution to the inhomogeneous equation is

$$y(t) = y_p(t) + C_1y_1(t) + C_2y_2(t).$$

Method of Undetermined Coefficients

If $f(t)$ has a form which is replicated under differentiation, then look for a particular solution of the same general form as the forcing term.
Exponential Forcing Term
\[ y'' + py' + qy = Ce^{bt} \]
- Example: \[ y'' + 3y' + 2y = 4e^{-3t} \]
- Try \( y_p(t) = ae^{-3t}; a \) to be determined.
- Particular solution: \( y_p(t) = 2e^{-3t} \).
- Homogeneous equation: \( y'' + 3y' + 2y = 0 \).
- Fundamental set of solutions: \( e^{-2t} \) & \( e^{-t} \).
- General solution to the inhomogeneous equation:
  \[ y(t) = 2e^{-3t} + C_1e^{-t} + C_2e^{-2t}. \]

Trigonometric Forcing Term
\[ y'' + py' + qy = A \cos \omega t + B \sin \omega t \]
- Example: \[ y'' + 4y' + 5y = 4 \cos 2t - 3 \sin 2t \]
- Try \( y_p(t) = a \cos 2t + b \sin 2t \)
- Particular solution: \( y_p(t) = \frac{28 \cos 2t + 29 \sin 2t}{65} \).
- Homogeneous equation: \( y'' + 4y' + 5y = 0 \)
- Fund. set of sol'n's: \( e^{-2t} \cos t \) & \( e^{-2t} \sin t \).
- General solution to the inhomogeneous equation:
  \[ y(t) = \frac{28 \cos 2t + 29 \sin 2t}{65} + e^{-2t}[C_1 \cos t + C_2 \sin t]. \]

Complex Method
\[ x'' + px' + qx = A \cos \omega t \text{ or } y'' + py' + qy = A \sin \omega t. \]
- Solve \( z'' + pz' + qz = Ae^{i\omega t} \).
- Try \( z(t) = ae^{i\omega t} \).
- Then
  \[ x_p(t) = \Re(z(t)) \text{ and } y_p(t) = \Im(z(t)). \]
**Example**

\[ x'' + 4x' + 5x = 4 \cos 2t \]

- Solve \( x'' + 4x' + 5x = 4e^{2t} \).
- Try \( z(t) = ae^{2t} \).
- Particular solution: \( z(t) = (4 - 32i)e^{2t}/65 \).
- Particular solution to the real equation:
  \[
  x_p(t) = \text{Re}(z(t)) \\
  = \frac{4 \cos 2t + 32 \sin 2t}{65}.
  \]

**Polynomial Forcing Term**

\[ y'' + py' + qy = P(t) \]

- Example: \( y'' - 3y' + 2y = 1 - 4t \).
- Try \( y(t) = a + bt \).
- Particular solution: \( y(t) = -5 - 2t \).
- General solution
  \[
  y(t) = -5 - 2t + C_1e^t + C_2e^{2t}.
  \]

**Exceptional Cases**

- Example: \( y'' - 3y' + 2y = 3e^t \).
- Try \( y(t) = ae^t \)
  - The method does not work because \( e^t \) is a solution to the associated homogeneous equation.
- Try \( y(t) = ate^t \)
  - Particular solution: \( y_p(t) = -3te^t \).
  - General solution: \( y(t) = -3te^t + C_1e^t + C_2e^{2t} \).
  - If the suggested particular solution does not work, multiply it by \( t \) and try again.
Combination Forcing Term

Example \( y'' + 5y' + 6y = 2e^{2t} - 5 \cos t \)
- Solve
  \[
  \begin{align*}
  y''_1 + 5y'_1 + 6y_1 &= 2e^{2t} \\
  y''_2 + 5y'_2 + 6y_2 &= -5 \cos t 
  \end{align*}
  \]
- Set \( y(t) = y_1(t) + y_2(t) \).

Forced Harmonic Motion

Assume an oscillatory forcing term:

\[
y'' + 2cy' + \omega_0^2 y = A \cos \omega t \]
- \( A \) is the forcing amplitude
- \( \omega \) is the forcing frequency
- \( \omega_0 \) is the natural frequency.
- \( c \) is the damping constant.

Forced Undamped Motion

\[
y'' + \omega_0^2 y = A \cos \omega t \]
- Homogeneous equation
  \[
y'' + \omega_0^2 y = 0
  \]
- General solution
  \[
y(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t.
  \]
- If \( \omega = \omega_0 \) we have an exceptional case.
• \( \omega \neq \omega_0 \)

\[ y'' + \omega_0^2 y = A \cos \omega t \]

• Look for a particular solution of the form

\[ x_p(t) = a \cos \omega t + b \sin \omega t. \]

• We find

\[ x_p(t) = \frac{A}{\omega_0^2 - \omega^2} \cos \omega t. \]

• \( \omega \neq \omega_0 \)

• General solution

\[ x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t. \]

• Initial conditions \( x(0) = x'(0) = 0 \) ⇒

\[ x(t) = \frac{A}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t]. \]

• Example: \( \omega_0 = 9, \omega = 8, A = \omega_0^2 - \omega^2 = 17. \)

\[ x(t) = \cos 9t - \cos 8t. \]

• \( \omega \neq \omega_0 \)

• Set \( \tau = \frac{\omega_0 + \omega}{2} \) and \( \delta = \frac{\omega_0 - \omega}{2} \).

\[ \Rightarrow \omega = \tau - \delta \text{ and } \omega_0 = \tau + \delta, \text{ and} \]

\[ x(t) = \frac{A}{\omega_0^2 - \omega^2} [\cos \omega t - \cos \omega_0 t]. \]

\[ \Rightarrow \frac{A}{2 \omega_0} \sin \delta t \sin \tau t. \]
• $\omega \neq \omega_0$
  - Example:
    
    \[ \omega = 8.5 \quad \text{and} \quad \delta = 0.5. \]
  
  - Envelope: Slow oscillation with frequency $\delta$,
    \[ \pm \frac{A \sin \delta t}{2\delta}. \]
  
  - Fast oscillation with frequency $\omega$ and varying amplitude.
  
  - Beats.

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• $\omega = \omega_0$
  
  \[ y'' + \omega_0^2 y = A \cos \omega_0 t. \]
  
  - We have an exceptional case. Try
    \[ x_p(t) = t[a \cos \omega t + b \sin \omega t]. \]
  
  - We find
    \[ x_p(t) = \frac{A}{2\omega_0} t \sin \omega_0 t. \]
  
  - General solution
    \[ x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t + \frac{A}{2\omega_0} t \sin \omega_0 t. \]

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• $\omega = \omega_0$
  
  - Initial conditions $x(0) = x'(0) = 0$ \Rightarrow
    \[ x(t) = \frac{A}{2\omega_0} t \sin \omega_0 t. \]
  
  - Example: $\omega_0 = 5$, and $A = 2\omega_0 = 10$.
    \[ x(t) = t \sin 5t. \]
  
  - Oscillation with increasing amplitude.
  
  - First example of resonance.
  
  - Forcing at the natural frequency can cause oscillations that grow out of control.