Forced Harmonic Motion

Assume an oscillatory forcing term:
\[ y'' + 2cy' + \omega_0^2 y = A \cos \omega t \]

- \( A \) is the forcing amplitude
- \( \omega \) is the forcing frequency
- \( \omega_0 \) is the natural frequency.
- \( c \) is the damping constant.

Forced Undamped Motion

\[ y'' + \omega_0^2 y = A \cos \omega t, \] where \( \omega \neq \omega_0 \)

- The solution with initial conditions \( x(0) = x'(0) = 0 \):
  \[ x(t) = \frac{A}{\omega_0^2 - \omega^2} \left[ \cos \omega t - \cos \omega_0 t \right] = \frac{A \sin \delta t}{2\omega_0} \sin \omega t, \]
  where \( \sigma = \frac{\omega_0 + \omega}{2} \) and \( \delta = \frac{\omega_0 - \omega}{2} \).
- This is a fast oscillation at frequency \( \sigma \), with amplitude oscillating slowly with frequency \( \delta \).
- This phenomenon is called “beats.”
Forced Undamped Motion (cont.)

\[ y'' + \omega_0^2 y = A \cos \omega_0 t, \quad \text{where} \quad \omega = \omega_0 \]

- An exceptional case.
- Solution with initial conditions \( x(0) = x'(0) = 0 \):
  \[ x_p(t) = \frac{A}{2\omega_0} \sin \omega_0 t. \]
  - The output is an oscillation with increasing amplitude.
  - First example of resonance.
  - Forcing at the natural frequency can cause oscillations that grow out of control.

Forced, Damped Harmonic Motion

\[ x'' + 2\xi x' + \omega_0^2 x = A \cos \omega t \]

Use the complex method.
- Solve \( z'' + 2\xi z' + \omega_0^2 z = Ae^{i\omega t} \).
  - We try \( z(t) = ae^{i\omega t} \) and get
    \[ z'' + 2\xi z' + \omega_0^2 z = [(i\omega)^2 + 2\xi(i\omega) + \omega_0^2]ae^{i\omega t} = P(i\omega)\]
  - where \( P(\lambda) = \lambda^2 + 2\xi \lambda + \omega_0^2 \) is the characteristic polynomial.
- The complex solution is \( z(t) = \frac{1}{P(i\omega)} Ae^{i\omega t} \).
- The real solution is \( x_p(t) = \text{Re}(z(t)) \).

Example

\[ x'' + 5x' + 4x = 50 \cos 3t \]

- \( P(\lambda) = \lambda^2 + 5\lambda + 4 \).
- \( P(i\omega) = P(3i) = -5 + 15i \)
- \( z(t) = \frac{1}{P(i\omega)} \cdot 50e^{3it} \)
  \[ = -[(\cos 3t - 3 \sin 3t) + i(\sin 3t + 3 \cos 3t)] \]
- \( x_p(t) = \text{Re}(z(t)) = 3 \sin 3t - \cos 3t \).
The Transfer Function

- The complex solution is
  \[ z(t) = \frac{1}{P(i\omega)} Ae^{i\omega t} = H(i\omega) Ae^{i\omega t}, \]
  where \( H(i\omega) = \frac{1}{P(i\omega)} \) is called the transfer function.
- We will use complex polar coordinates to write
  \[ H(i\omega) = G(\omega)e^{-i\phi(\omega)}, \]
  where \( G(\omega) = |H(i\omega)| \) is the called the gain and \( \phi(\omega) \) is called the phase shift.

The Gain and Phase Shift

- If \( P(\lambda) = \lambda^2 + 2c\lambda + \omega_0^2 \) is the characteristic polynomial, then \( P(i\omega) = Re^{i\phi} \), where
  \[ R = \sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}, \] and
  \[ \phi = \arccot \left( \frac{\omega_0^2 - \omega^2}{2c\omega} \right). \]
- The transfer function is
  \[ H(i\omega) = \frac{1}{P(i\omega)} \frac{1}{R} e^{-i\phi} = G(\omega)e^{-i\phi}. \]
- The gain \( G(\omega) = \frac{1}{R} = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}}. \)

The complex particular solution is
\[ z(t) = H(i\omega) Ae^{i\omega t} = G(\omega)e^{-i\phi} \cdot Ae^{i\omega t} = G(\omega)Ae^{i(\omega t-\phi)}. \]
- The real particular solution is
  \[ x_p(t) = \text{Re}(z(t)) = G(\omega)A\cos(\omega t - \phi). \]
- The amplitude of \( x_p \) is \( G(\omega)A \), and the phase is \( \phi \).
• The general solution is
  \[ x(t) = x_p(t) + x_h(t) = G(\omega)A \cos(\omega t - \phi) + x_h(t), \]
  where \( x_h(t) \) is the general solution of the homogeneous equation.
• \( x_h(t) \to 0 \) as \( t \) increases, so \( x_h \) is called the transient term.
• \( x_p(t) = G(\omega)A \cos(\omega t - \phi) \) is called the steady-state solution.

Example

\[ x'' + 5x' + 4x = 50 \cos 3t \]

• \( G(\omega) = \frac{1}{\sqrt{(4 - \omega^2)^2 + 25\omega^2}} \) and
  \[ \phi = \arccot \left( \frac{4 - \omega^2}{5\omega} \right). \]
• With \( \omega = 3 \),
  \[ G(3) = \frac{1}{5\sqrt{10}} \approx 0.0632 \]
  \[ \phi = \arccot(-3/5) \approx 2.1112. \]
• SS solution \( x_p(t) = G(3)A \cos(3t - \phi). \)

The Steady-State Solution

\[ x_p(t) = G(\omega)A \cos(\omega t - \phi). \]

• The forcing function is \( A \cos \omega t \).
• Properties of the steady-state response:
  • It is oscillatory at the driving frequency.
  • The amplitude is the product of the gain, \( G(\omega) \), and the amplitude of the forcing function.
  • It has a phase shift of \( \phi \) with respect to the forcing function.
The Gain

\[ G(\omega) = \frac{1}{\sqrt{(\omega_0^2 - \omega^2)^2 + 4c^2\omega^2}} \]

Set \( \omega = s\omega_0 \) and \( c = D\omega_0/2 \) (or \( s = \omega/\omega_0 \) and \( D = 2c/\omega_0 \)). Then

\[ G(\omega) = \frac{1}{\omega_0} \frac{1}{\sqrt{(1 - s^2)^2 + D^2s^2}} \]