Math 211

Lecture #36

Nonlinear Systems

November 20, 2002

Interacting Species

- Two species with populations $x_1$ & $x_2$.
- Interaction between the species can be helpful or detrimental.
- Basic model
  \[
  x_1' = r_1 x_1 \\
  x_2' = r_2 x_2
  \]
  - $r_1$ & $r_2$ are the reproductive rates.

Reproductive Rates

- If $x_2 = 0$ the reproductive rate for $x_1$ is
  \[
  r_1 = a_1 - b_1 x_1.
  \]
  - $a_1 > 0 \Rightarrow$ natural growth.
  - $a_1 < 0 \Rightarrow$ natural decline.
  - $b_1 = 0$ Malthusian growth.
  - $b_1 > 0$ logistic growth.
• If $x_2 > 0$ the reproductive rate for $x_1$ is
  \[ r_1 = a_1 - b_1 x_1 + c_1 x_2. \]
• $c_1 > 0 \Rightarrow$ interaction is helpful to $x_1$.
• $c_1 < 0 \Rightarrow$ interaction is detrimental to $x_1$.
• The reproductive rate for $x_2$ is
  \[ r_2 = a_2 - b_2 x_2 + c_2 x_1. \]
• The model for interacting species is
  \[
  x_1' = (a_1 - b_1 x_1 + c_1 x_2)x_1 \\
  x_2' = (a_2 - b_2 x_2 + c_2 x_1)x_2
  \]

Predator Prey Model

Rabbits & foxes, fish & sharks, and cottony cushion scale insect & ladybird beetle.
• $F = \text{fish} \& S = \text{sharks}$.
  \[
  F' = (a - bS)F \\
  S' = (-c + dF)S
  \]
  or
  \[
  F' = (a - eF - bS)F \\
  S' = (-c + dF)S
  \]
  $a = 3, b = 3, c = 1, d = 3, e = 3$.

Competing Species

Cattle and sheep.
• $x_1$ and $x_2$ competing for resources.
  \[
  x_1' = (a_1 - b_1 x_1 + c_1 x_2)x_1 \\
  x_2' = (a_2 - b_2 x_2 + c_2 x_1)x_2
  \]
• $a_2 > 0, b_1 > 0, \& c_1 < 0$
• Example:
  \[
  x' = (5 - 2x - y)x \\
  y' = (7 - 2x - 3y)y
  \]
Linearization
The principal idea of differential calculus:

- Approximate nonlinear mathematical objects by linear ones.
- Example: Approximate the function $f(y)$ near $y_0$ by a linear function.
  
  $$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$
  where $\lim_{h \to 0} \frac{R(h)}{h} = 0$.

- The linear function is $L(h) = f(y_0) + f'(y_0)h$.

Linearization of an ODE

- $y' = f(y)$
- Assume $f(y_0) = 0$ and $f'(y_0) \neq 0$.
- Set $y = y_0 + u$. Get
  
  $$u' = f(y_0 + u) = f'(y_0)u + R(u)$$

- Approximate by the linear differential equation
  
  $$\tilde{u}' = f'(y_0)\tilde{u}$$

- If $f'(y_0) \neq 0$ the equilibrium point of the linearization at 0 has the same stability properties as that of the nonlinear equation at $y_0$.
  
  - $f'(y_0) > 0 \Rightarrow y_0$ is unstable.
  - $f'(y_0) < 0 \Rightarrow y_0$ is asymptotically stable.

  - We can solve the linearization explicitly.
Linearization of a Planar System

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

- Assume \((x_0, y_0)\) is an equilibrium point, so
  \[ f(x_0, y_0) = g(x_0, y_0) = 0 \]

We have by Taylor's theorem

\[
\begin{align*}
  f(x_0 + u, y_0 + v) &= \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v) \\
  g(x_0 + u, y_0 + v) &= \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)
\end{align*}
\]

where \( \frac{R_f(u, v)}{\sqrt{u^2 + v^2}} \to 0 \) and \( \frac{R_g(u, v)}{\sqrt{u^2 + v^2}} \to 0 \)

- Set \( x = x_0 + u \) and \( y = y_0 + v \). The system becomes

\[
\begin{align*}
  u' &= \frac{\partial f}{\partial x}(x_0, y_0)u + \frac{\partial f}{\partial y}(x_0, y_0)v + R_f(u, v) \\
  v' &= \frac{\partial g}{\partial x}(x_0, y_0)u + \frac{\partial g}{\partial y}(x_0, y_0)v + R_g(u, v)
\end{align*}
\]
Linearization at \((x_0, y_0)\)

\[
\begin{align*}
\dot{u}' &= \frac{\partial f}{\partial x}(x_0, y_0)\dot{u} + \frac{\partial f}{\partial y}(x_0, y_0)\dot{v} \\
\dot{v}' &= \frac{\partial g}{\partial x}(x_0, y_0)\dot{u} + \frac{\partial g}{\partial y}(x_0, y_0)\dot{v}
\end{align*}
\]

- This is a linear system.
- We can solve it explicitly.
- Does it give information about the original nonlinear system?

Matrix Form of the Linearization

Set \(u = (\dot{u}, \dot{v})^T\) and introduce the Jacobian matrix

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
\]

- The linearization becomes

\[
u' = Ju.
\]

Theorem: Consider the planar system

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

where \(f\) and \(g\) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.
Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
- All occupy large open subsets of the trace-determinant plane.
- Nongeneric types
  - Center and others. Occupy pieces of the boundaries.

Examples

- Predator prey
- Competing species
- Center
  \[
  x' = y + ax(x^2 + y^2) \\
  y' = -x + ay(x^2 + y^2)
  \]