Linearization

The principal idea of differential calculus:
- Approximate nonlinear mathematical objects by linear ones.
- Example: Approximate the function $f(y)$ near $y_0$ by a linear function.
  
  $f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$

  where $\lim_{h \to 0} \frac{R(h)}{h} = 0$.

- The linear function is $L(h) = f(y_0) + f'(y_0)h$.

Linearization of a Planar System

$x' = f(x, y)$
$y' = g(x, y)$

- $(x_0, y_0)$ is an equilibrium point so
  
  $f(x_0, y_0) = g(x_0, y_0) = 0$
Taylor’s Theorem

- We have by Taylor’s theorem

\[ f(x_0 + u, y_0 + v) = \partial f \partial x(x_0, y_0) \cdot u + \partial f \partial y(x_0, y_0) \cdot v + R_f(u, v) \]

\[ g(x_0 + u, y_0 + v) = \partial g \partial x(x_0, y_0) \cdot u + \partial g \partial y(x_0, y_0) \cdot v + R_g(u, v) \]

where \( R_f(u, v) \sqrt{u^2 + v^2} \to 0 \) and \( R_g(u, v) \sqrt{u^2 + v^2} \to 0 \)

The Linearization at \((x_0, y_0)\)

- Set \( x = x_0 + u \) and \( y = y_0 + v \). The system becomes

\[ u' = \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v) \]

\[ v' = \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v) \]

- The linearization is defined to be

\[ \tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v} \]

\[ \tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v} \]

Matrix Form of the Linearization

Set \( u = (\tilde{u}, \tilde{v})^T \) and introduce the Jacobian matrix

\[ J = \begin{pmatrix} \frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\ \frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0) \end{pmatrix} \]

- The linearization becomes

\[ u' = J u \]

- The behavior of solutions to the linearization is determined by the eigenvalues of the Jacobian.
Properties of the Linearization

- The linearization gives us information about the original system.

**Theorem:** Consider the planar system

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

where \(f\) and \(g\) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.

Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
- All occupy large open subsets of the trace-determinant plane.

Nongeneric types
- Center and others, which occupy pieces of the boundaries between the generic points.

Examples

- Center.
  \[
  \begin{align*}
x' &= y + \alpha x(x^2 + y^2) \\
y' &= -x + \alpha y(x^2 + y^2)
\end{align*}
\]
- \(\alpha > 0 \implies (0, 0)^T\) is unstable.
- \(\alpha < 0 \implies (0, 0)^T\) is a sink.
- Competing species.
- Default system in pplane.
Linear Analysis of Equilibrium Points

- Provides a good qualitative picture of how solutions behave near generic equilibrium points.
- Provides limited qualitative information about the solutions near nongeneric equilibrium points.
  - A linear center could be a spiral source or a spiral sink.
- Provides no information about the global behavior of solutions to nonlinear systems.

Higher Dimensional Systems

Autonomous equation \( y' = f(y) \).

- \( y = (y_1, y_2, \ldots, y_n)^T \), \( y_0 \) is an equilibrium point.
- \( f(y) = (f_1(y), f_2(y), \ldots, f_n(y))^T \)
- \( J \) is the Jacobian matrix
- \( f(y_0 + u) = J(y_0)u + R(u) \) where \( \lim_{|u| \to 0} \frac{R(u)}{|u|} = 0 \).
- Set \( y = y_0 + u \). The system becomes
  \( u' = J(y_0)u + R(u) \).
- The linearization is \( u' = J(y_0)u \).

Theorem: Suppose that \( y_0 \) is an equilibrium point for \( y' = f(y) \). Let \( J \) be the Jacobian of \( f \) at \( y_0 \).

1. Suppose that the real part of every eigenvalue of \( J \) is negative. Then \( y_0 \) is an asymptotically stable equilibrium point.
2. Suppose that \( J \) has at least one eigenvalue with positive real part. Then \( y_0 \) is an unstable equilibrium point.
Example

\[ x' = -2x - 4y + 2xy \]
\[ y' = x - 6y + x^2 - y^2 \]

- The origin \((0, 0)\) is an equilibrium point.
- The Jacobian has one eigenvalue, \(\lambda = -4\), of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem \(\Rightarrow\) the origin is a sink.