Linearization

The principal idea of differential calculus:

• Approximate nonlinear mathematical objects by linear ones.

• Example: Approximate the function $f(y)$ near $y_0$ by a linear function.

$$f(y_0 + h) = f(y_0) + f'(y_0)h + R(h)$$

where

$$\lim_{h \to 0} \frac{R(h)}{h} = 0.$$ 

♦ The linear function is $L(h) = f(y_0) + f'(y_0)h$. 
Linearization of a Planar System

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

- \((x_0, y_0)\) is an equilibrium point so

\[ f(x_0, y_0) = g(x_0, y_0) = 0 \]
Taylor’s Theorem

- We have by Taylor’s theorem

\[ f(x_0 + u, y_0 + v) = \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v) \]

\[ g(x_0 + u, y_0 + v) = \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v) \]

where \( \frac{R_f(u, v)}{\sqrt{u^2 + v^2}} \rightarrow 0 \) and \( \frac{R_g(u, v)}{\sqrt{u^2 + v^2}} \rightarrow 0 \)
The Linearization at \((x_0, y_0)\)

- Set \(x = x_0 + u\) and \(y = y_0 + v\). The system becomes

\[
\begin{align*}
    u' &= \frac{\partial f}{\partial x}(x_0, y_0) \cdot u + \frac{\partial f}{\partial y}(x_0, y_0) \cdot v + R_f(u, v) \\
    v' &= \frac{\partial g}{\partial x}(x_0, y_0) \cdot u + \frac{\partial g}{\partial y}(x_0, y_0) \cdot v + R_g(u, v)
\end{align*}
\]

- The linearization is defined to be

\[
\begin{align*}
    \tilde{u}' &= \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v} \\
    \tilde{v}' &= \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v}
\end{align*}
\]
Matrix Form of the Linearization

Set $u = (\tilde{u}, \tilde{v})^T$ and introduce the \textit{Jacobian matrix}

$$J = \begin{pmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}$$

- The linearization becomes

$$u' = Ju.$$

- The behavior of solutions to the linearization is determined by the eigenvalues of the Jacobian.
Properties of the Linearization

- The linearization gives us information about the original system.

**Theorem:** Consider the planar system

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y)
\end{align*}
\]

where \( f \) and \( g \) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.
Generic Equilibrium Points

- Saddle, nodal source, nodal sink, spiral source, and spiral sink.
  - All occupy large open subsets of the trace-determinant plane.

- Nongeneric types
  - Center and others, which occupy pieces of the boundaries between the generic points.
Examples

• Center.

\[ x' = y + \alpha x(x^2 + y^2) \]
\[ y' = -x + \alpha y(x^2 + y^2) \]

\[ \alpha > 0 \implies (0, 0)^T \text{ is unstable.} \]
\[ \alpha < 0 \implies (0, 0)^T \text{ is a sink.} \]

• Competing species.

• Default system in pplane.
Linear Analysis of Equilibrium Points

- Provides a good qualitative picture of how solutions behave near generic equilibrium points.
- Provides limited qualitative information about the solutions near nongeneric equilibrium points.
  - A linear center could be a spiral source or a spiral sink.
- Provides no information about the global behavior of solutions to nonlinear systems.
Higher Dimensional Systems

Autonomous equation $y' = f(y)$.

- $y = (y_1, y_2, \cdots, y_n)^T$, $y_0$ is an equilibrium point.
- $f(y) = (f_1(y), f_2(y), \cdots, f_n(y))^T$
- $J$ is the Jacobian matrix
- $f(y_0 + u) = J(y_0)u + R(u)$ where $\lim_{u \to 0} \frac{R(u)}{|u|} = 0$.
- Set $y = y_0 + u$. The system becomes

$$u' = J(y_0)u + R(u).$$

- The linearization is $u' = J(y_0)u$. 

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Theorem: Suppose that $y_0$ is an equilibrium point for $y' = f(y)$. Let $J$ be the Jacobian of $f$ at $y_0$.

1. Suppose that the real part of every eigenvalue of $J$ is negative. Then $y_0$ is an asymptotically stable equilibrium point.

2. Suppose that $J$ has at least one eigenvalue with positive real part. Then $y_0$ is an unstable equilibrium point.
Example

\[ x' = -2x - 4y + 2xy \]
\[ y' = x - 6y + x^2 - y^2 \]

- The origin \((0, 0)\) is an equilibrium point.
- The Jacobian has one eigenvalue, \(\lambda = -4\), of algebraic multiplicity 2.
- **First theorem** does not apply.
- **Second theorem** \(\Rightarrow\) the origin is a sink.