Higher Dimensional Systems

Autonomous equation $y' = f(y)$.

- $y = (y_1, y_2, \cdots, y_n)^T$, $y_0$ is an equilibrium point.
- $f(y) = (f_1(y), f_2(y), \cdots, f_n(y))^T$
- $J$ is the Jacobian matrix
- $f(y_0 + u) = J(y_0)u + R(u)$ where $\lim_{|u| \to 0} \frac{R(u)}{|u|} = 0$.
- Set $y = y_0 + u$. The system becomes
  
  $$u' = J(y_0)u + R(u).$$

- The linearization is $u' = J(y_0)u$.

Linearization of a Planar System

$$x' = f(x, y) \quad f(x_0, y_0) = 0.$$  
$$y' = g(x, y) \quad g(x_0, y_0) = 0.$$  

- The linearization at $(x_0, y_0)$ is
  
  $$\tilde{u}' = \frac{\partial f}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial f}{\partial y}(x_0, y_0) \cdot \tilde{v}$$
  
  $$\tilde{v}' = \frac{\partial g}{\partial x}(x_0, y_0) \cdot \tilde{u} + \frac{\partial g}{\partial y}(x_0, y_0) \cdot \tilde{v}.$$
The Jacobian

Set \( u = (\tilde{u}, \tilde{v})^T \). The Jacobian matrix is

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
\]

- The linearization becomes
  \( u' = Ju \).

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Theorem: Consider the planar system

\[
x' = f(x, y) \\
y' = g(x, y)
\]

where \( f \) and \( g \) are continuously differentiable. Suppose that \((x_0, y_0)\) is an equilibrium point. If the linearization at \((x_0, y_0)\) has a generic equilibrium point at the origin, then the equilibrium point at \((x_0, y_0)\) is of the same type.

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Theorem: Suppose that \( y_0 \) is an equilibrium point for \( y' = f(y) \). Let \( J \) be the Jacobian of \( f \) at \( y_0 \).

1. Suppose that the real part of every eigenvalue of \( J \) is negative. Then \( y_0 \) is an asymptotically stable equilibrium point.
2. Suppose that \( J \) has at least one eigenvalue with positive real part. Then \( y_0 \) is an unstable equilibrium point.
Example

\[ x' = -2x - 4y + 2xy \]
\[ y' = x - 6y + x^2 - y^2 \]

- The origin (0, 0) is an equilibrium point.
- The Jacobian has one eigenvalue, \( \lambda = -4 \), of algebraic multiplicity 2.
- First theorem does not apply.
- Second theorem \( \Rightarrow \) the origin is a sink.

The Lorenz System

\[ x' = -ax + ay \]
\[ y' = rx - y - xz \]
\[ z' = -bz + xy \]

- Equilibrium points.
  - \( (r \leq 1) \) (0, 0, 0)
  - \( (r > 1) \) Set \( s = \sqrt{b(r - 1)} \). The equilibrium points are (0, 0, 0), and \( c^\pm = (\pm s, \pm s, r - 1) \).

The Jacobian is

\[
J = \begin{pmatrix}
-a & a & 0 \\
r - z & -1 & -x \\
y & x & -b \\
\end{pmatrix}
\]

- Use \( a = 10 \) and \( b = 8/3 \).
- \( (0, 0, 0) \)
- If \( r < 1 \) \( (0, 0, 0) \) is asymptotically stable.
- If \( r > 1 \) \( (0, 0, 0) \) is unstable.
c^+ and c^-
  - For 1 < r < 470/19 \approx 24.74, c^+ and c^- are asymptotically stable.
  - For r > 470/19 \approx 24.74, c^+ and c^- are unstable.
- As r varies the Lorenz system displays a wide variety of behaviors.
  - For r = 28 we have Lorenz's strange attractor.
  - For r = 100 there is a periodic attractor.
  - For r = 200 there is another strange attractor.

Invariant Sets
Definition: A set S is (positively) invariant for the system y' = f(y) if y(0) = y_0 \in S implies that y(t) \in S for all t \geq 0.
- Examples:
  - An equilibrium point.
  - Any solution curve.

Example — Competing Species
\[
x' = (5 - 2x - y)x \\
y' = (7 - 2x - 3y)y
\]
- The positive x- and y-axes are invariant.
- The positive quadrant is invariant.
- Populations should remain nonnegative.
- The set S = \{(x, y) \mid 0 < x < 3, \ 0 < y < 3\} is positively invariant.
Nullclines

\[ x' = f(x, y) \]
\[ y' = g(x, y) \]

**Definition:** The \( x \)-nullcline is the set defined by \( f(x, y) = 0 \). The \( y \)-nullcline is the set defined by \( g(x, y) = 0 \).

- Along the \( x \)-nullcline the vector field points up or down.
- Along the \( y \)-nullcline the vector field points left or right.
- The nullclines intersect at the equilibrium points.

Competing Species

\[ x' = (5 - 2x - y)x \]
\[ y' = (7 - 2x - 3y)y \]

- \( x \)-nullcline: two lines \( x = 0 \) and \( 2x + y = 5 \).
- \( y \)-nullcline: two lines \( y = 0 \) and \( 2x + 3y = 7 \).
- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant \( \to (2, 1) \) as \( t \to \infty \).