Math 211

Lecture #39

Invariant Sets

November 27, 2002
Review of Methods

Linearization at an equilibrium point

• $y' = f(y)$ has an equilibrium point at $y_0$.

• The linearization $u' = J(y_0)u$ has an equilibrium point at $u = 0$.

• The linearization can sometimes predict the behavior of solutions to the nonlinear system near the equilibrium point.

♦ The linearization gives only local information.

♦ We need ways to discover the global behavior of solutions.
Invariant Sets

**Definition:** A set $S$ is *(positively) invariant* for the system $y' = f(y)$ if $y(0) = y_0 \in S$ implies that $y(t) \in S$ for all $t \geq 0$.

- **Examples:**
  - An equilibrium point.
  - Any solution curve.
Example — Competing Species

\[ x' = (5 - 2x - y)x \]
\[ y' = (7 - 2x - 3y)y \]

- The positive \( x \)- and \( y \)-axes are invariant.
- The positive quadrant is invariant.
  - Populations should remain nonnegative.
- The set \( S = \{(x, y) \mid 0 < x < 3, \ 0 < y < 3\} \) is positively invariant.
**Nullclines**

\[
x' = f(x, y)
\]

\[
y' = g(x, y)
\]

**Definition:** The \textit{x-nullcline} is the set defined by \( f(x, y) = 0 \). The \textit{y-nullcline} is the set defined by \( g(x, y) = 0 \).

- Along the \textit{x-nullcline} the vector field points up or down.
- Along the \textit{y-nullcline} the vector field points left or right.
- The nullclines intersect at the equilibrium points.
Competing Species

\[ x' = (5 - 2x - y)x \]
\[ y' = (7 - 2x - 3y)y \]

- **x-nullcline:** two lines \( x = 0 \) and \( 2x + y = 5 \).
- **y-nullcline:** two lines \( y = 0 \) and \( 2x + 3y = 7 \).
- Two of the four regions in the positive quadrant defined by the nullclines are positively invariant.
- This information allows us to predict that all solutions in the positive quadrant \( \rightarrow (2, 1) \) as \( t \rightarrow \infty \).
Competing Species – 2nd Example

\[ x' = (1 - x - y)x \]
\[ y' = (4 - 7x - 3y)y \]

- The axes are invariant. The positive quadrant is invariant.
- The equilibrium point at \((1/4, 3/4)\) is a saddle point.
- Almost all solutions go to one of the nodal sinks \((0, 4/3)\) or \((1, 0)\).
**Definition:** The *basin of attraction* of a sink $y_0$ consists of all points $y$ such that the solution starting at $y$ approaches $y_0$ as $t \to \infty$.

- In the example, the basins of attraction of the two sinks are separated by the stable orbits of the saddle point.
- The stable and unstable orbits of a saddle point are called *separatrices*. (Separatrices is the plural of separatrix.)
Summary

• Sometimes the understanding of invariant sets can help us understand the long term behavior of all solutions.

• Nullclines can sometimes help us find informative invariant sets.

• None of this helps us understand the predator-prey system.