Math 211

Lecture #40
Long Term Behavior of Planar Systems

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Example

\[
\begin{align*}
x' &= (2 - x - y)x \\
y' &= (3 - x^2 - y^2)y
\end{align*}
\]

- The axes are invariant. The positive quadrant is invariant.
- \(x\)-nullcline: two lines \(x = 0\) and \(x + y = 2\).
- \(y\)-nullcline: line \(y = 0\) and circle \(x^2 + y^2 = 3\).
- The equilibrium points are \((0, 0)\), \((0, \sqrt{3})\), \((2, 0)\), \((a, b)\), and \((b, a)\), where \(a = 1 - \frac{1}{\sqrt{2}}\) and \(b = 2 - a\).

Example (cont.)

- Classification: \((0, 0)\) is a nodal source, \((0, \sqrt{3})\) and \((b, a)\) are saddles, \((2, 0)\) and \((a, b)\) are nodal sinks.
- The nullclines divide the positive quadrant into 5 regions, of which 3 are invariant.
- Almost all solution curves in the positive quadrant are attracted to the nodal sinks \((2, 0)\) and \((a, b)\).
- The basins of attraction of \((2, 0)\) and \((a, b)\) are separated by the stable solution curves for the saddle at \((b, a)\).
Basic Question about a System $y' = f(y)$

- What happens to all solutions as $t \to \infty$?
- What are the possibilities as $t \to \infty$?
  - Is there a small list of all possibilities?
  - We need a definitive notion of what a “possibility” is.

Limit Sets

Definition: The (forward) limit set of the solution $y(t)$ that starts at $y_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(y_0)$.

- $x \in \omega(y_0)$ if there is a sequence $t_k \to \infty$ such that $y(t_k) \to x$.

- What kinds of sets can be limit sets?
  - The empty set.
  - Equilibrium points.
  - Periodic solution curves.

- Is there a small list of all possible limit sets?

Properties of Limit Sets

Theorem: Suppose that the system $y' = f(y)$ is defined in the set $U$.

1. If the solution curve starting at $y_0$ stays in a bounded subset of $U$, then the limit set $\omega(y_0)$ is not empty.
2. Any limit set is both positively and negatively invariant.
Example

\[ x' = 5y + x(9 - x^2 - y^2) \]
\[ y' = -5x + y(9 - x^2 - y^2) \]

- The origin is a spiral source.
- In polar coordinates the system is
  \[ r' = r(9 - r^2) \]
  \[ \theta' = -5 \]

- All solution curves approach the circle \( x^2 + y^2 = 9 \).
- The circle \( x^2 + y^2 = 9 \) is a solution curve.

Limit Cycle

Definition: A limit cycle is a closed solution curve which is the limit set of nearby solution curves. If the solution curves spiral into the limit cycle as \( t \to \infty \), it is a attracting limit cycle. If they spiral into the limit cycle as \( t \to -\infty \), it is a repelling limit cycle.

- In the example the circle \( x^2 + y^2 = 9 \) is a limit cycle.

Types of Limit Set

- A limit cycle is a new type of phenomenon.
- However, the limit set is a periodic orbit, so the type of limit set is not new.
- We still have only two types of non-empty limits sets.
  - An equilibrium point.
  - A closed solution curve.
Example
\[ x' = (y + x/5)(1 - x^2) \]
\[ y' = -x(1 - y^2) \]

- The lines \( x = \pm 1 \) and \( y = \pm 1 \) are invariant.
- The unit square is invariant.
- The corners of the unit square are saddle points.
  - The lines \( x = \pm 1 \) and \( y = \pm 1 \) are separatrices.
- The origin is a spiral source.
- The limit set of any solution that starts in the unit square is the boundary of the unit square.

Planar Graph
Definition: A planar graph is a collection of points, called vertices, and non-intersecting curves, called edges, which connect the vertices. If the edges each have a direction the graph is said to be directed.

- The boundary of the unit square in the example is a directed planar graph.

Theorem: If \( S \) is a nonempty limit set of a solution of a planar system defined in a set \( U \subset \mathbb{R}^2 \), then \( S \) is one of the following:
- An equilibrium point.
- A closed solution curve.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.
These are called the Poincaré-Bendixson alternatives.