Math 211

Review for the Final Exam

December 7, 2003

The Final Exam

- The final will be comprehensive, covering material from the entire semester.
- The final will emphasize the material covered since the last exam.
- These slides will cover primarily the material covered since the last exam. They do not cover all of the material on the exam.
- Questions about any of the material of the course will be answered.

The Themes of the Course

- Modeling.
  - Population, finance, mixing, motion, vibrating spring, electrical circuits, . . .
- Exact solutions.
  - Separable and linear equations in dimension 1.
- Linear equations in higher dimension.
  - Matrix algebra.
- Second order linear equations.
- Numerical solutions.
- Qualitative analysis.
Solving $x' = Ax$

- $A$ is an $n \times n$ matrix.
- Solution strategy: Look for a fundamental set of solutions, i.e., $n$ linearly independent solutions.
- The function $x(t) = e^{A}v$ solves the initial value problem $x' = Ax$ with $x(0) = v$.
- Refined strategy: Compute $e^{A}v$ for $n$ linearly independent vectors $v$.
- Computing $e^{A}v$ is hard except for specially chosen vectors $v$.

Procedure for Solving $x' = Ax$

- Find the eigenvalues of $A$ and their algebraic multiplicities.
- For each eigenvalue $\lambda$ with algebraic multiplicity $q$:
  - Find the smallest integer $k$ for which null($[A - \lambda I]^k$) has dimension $q$.
  - Find a basis for null($[A - \lambda I]^k$).
  - For each vector $v$ in the basis compute the solution $x(t) = e^{\lambda}v$.
- The set of all of these solutions is a fundamental set of solutions.
- Replace complex solutions with real solutions.

Solutions to Higher Order Equations

Homogenous linear equation with constant coefficients:

$$y'' + py' + qy = 0$$

- Look for exponential solutions $y(t) = e^{\lambda t}$.
- Characteristic polynomial: $\lambda^2 + p\lambda + q$.
- If $\lambda$ is a root of the characteristic polynomial then $y(t) = e^{\lambda t}$ is a solution.
Fundamental sets of solutions

• Two distinct real roots $\lambda_1$ and $\lambda_2$:
  \[ y_1(t) = e^{\lambda_1 t} \quad \text{and} \quad y_2(t) = e^{\lambda_2 t}. \]

• One real root $\lambda$ of multiplicity 2:
  \[ y_1(t) = e^{\lambda t} \quad \text{and} \quad y_2(t) = te^{\lambda t}. \]

• Complex conjugate roots $\lambda = \alpha \pm i\beta$:
  \[ y_1(t) = e^{\alpha t} \cos \beta t \quad \text{and} \quad y_2(t) = e^{\alpha t} \sin \beta t. \]

Inhomogeneous Equations

\[ y'' + Py' + Qy = f(t) \]

• If $y_p$ is a particular solution, the general solution is
  \[ y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t), \]
  where $y_1$ and $y_2$ are a fundamental set of solutions to the homogeneous equation.

• The method of undetermined coefficients finds a particular solution $y_p(t)$.
  • If the forcing term $f(t)$ has a form which is replicated under differentiation, look for a particular solution of the same general form as the forcing term.

Cases

• If $f(t) = C e^{\lambda t}$, try $y_p(t) = a e^{\lambda t}$.
• If $f(t) = A \cos \omega t + B \sin \omega t$, try $y_p(t) = a \cos \omega t + b \sin \omega t$.
  • Or try the complex method.
• If $f(t)$ is a polynomial of degree $n$, let $y_p$ be a polynomial of degree $n$.
• Exceptional cases: Multiply expected form of $y_p$ by $t$.
• Combination cases: Solve the equation in pieces.
**Harmonic Motion**

- **Spring:** \( y'' + \frac{\mu}{m}y' + \frac{k}{m}y = \frac{1}{m}F(t) \).
- **Circuit:** \( I'' + \frac{R}{L}I' + \frac{1}{LC}I = \frac{1}{L}E'(t) \).
- Essentially the same equation. Use
  \[ x'' + 2cx' + \omega_0^2x = f(t). \]
  - We call this the equation for harmonic motion.
  - \( \omega_0 \) is the natural frequency. \( c \) is the damping constant. \( f(t) \) is the forcing term.

**Unforced Harmonic Motion**

\[ x'' + 2cx' + \omega_0^2x = 0 \]

- Undamped: \( c = 0 \).
- Underdamped: \( 0 < c < \omega_0 \).
- Critically damped: \( c = \omega_0 \).
- Over damped: \( c > \omega_0 \).

**Forced Harmonic Motion**

\[ x'' + 2cx' + \omega_0^2x = A\cos\omega t \]

- \( A \) is the forcing amplitude and \( \omega \) is the forcing frequency.
- The general solution is \( x(t) = x_p(t) + x_h(t) \).
  - \( x_p \) is a particular solution. \( x_h \) is the general solution of the homogenous equation.
  - Undamped: \( c = 0 \).
  - \( \omega \neq \omega_0 \): Beats.
  - \( \omega = \omega_0 \): Resonance.
Forced, Damped Harmonic Motion
\[ x'' + 2\alpha x' + \omega_0^2 x = A \cos \omega t \]
\[ x(t) = x_p(t) + x_h(t). \]

- \( c > 0 \) implies that the roots have negative real part, so that \( x_h(t) \to 0 \) as \( t \) increases, so \( x_h \) is called the transient term.
- \( x_p(t) \) is called the steady-state solution. It has the form
  \[ x_p(t) = G(\omega) A \cos(\omega t - \phi) \]
- \( x_p \) is oscillatory at the driving frequency.
- The amplitude of \( x_p \) is the product of the gain, \( G(\omega) \), and the amplitude of the forcing function.
- \( x_p \) has a phase shift of \( \phi \) with respect to the forcing function.

Qualitative Analysis

- Existence and uniqueness.
- For an autonomous system \( x' = f(x) \), the basic question is, What happens to all solutions as \( t \to \infty \)?
- The easy cases: equilibrium points \( f(x_0) = 0 \) and equilibrium solutions \( x(t) = x_0 \).
- Local qualitative analysis: What happens as \( t \to \infty \) to all solutions that start near an equilibrium point \( x_0 \) ?
  - This is the question of stability.
- Global qualitative analysis: What happens to all solutions as \( t \to \infty \)?

Stability

Suppose the autonomous system \( x' = f(x) \) has an equilibrium point at \( x_0 \).
- \( x_0 \) is stable if every solution that starts close to \( x_0 \) stays close to \( x_0 \).
- \( x_0 \) is asymptotically stable if every solution that starts close to \( x_0 \) stays near \( x_0 \) and approaches \( x_0 \) as \( t \to \infty \).
- \( x_0 \) is called a sink.
- \( x_0 \) is unstable if there are solutions starting arbitrarily close to \( x_0 \) that move away from \( x_0 \).
Stability for $x' = Ax$

- $D = 2$: Trace-determinant plane.
- Generic equilibrium points: Nodal sources, spiral sources, nodal sinks, spiral sinks, saddles.
- Nongeneric equilibrium points: Centers, etc.

**Theorem:** Let $A$ be an $n \times n$ real matrix.

- Suppose the real part of every eigenvalue of $A$ is negative. Then 0 is an asymptotically stable equilibrium point for the system $x' = Ax$.
- Suppose $A$ has at least one eigenvalue with positive real part. Then 0 is an unstable equilibrium point for the system $x' = Ax$.

Stability for $x' = f(x)$

- Suppose that $x_0$ is an equilibrium point.
- The linearization at $x_0$ is the system $u' = J u$, where $J$ is the Jacobian matrix of $f$ at $x_0$.
- For the planar system $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$, the Jacobian is

\[
J = \begin{pmatrix}
\frac{\partial f}{\partial x}(x_0, y_0) & \frac{\partial f}{\partial y}(x_0, y_0) \\
\frac{\partial g}{\partial x}(x_0, y_0) & \frac{\partial g}{\partial y}(x_0, y_0)
\end{pmatrix}
\]

Theorem: Consider the planar system $\begin{cases} x' = f(x, y) \\ y' = g(x, y) \end{cases}$ where $f$ and $g$ are continuously differentiable. Suppose that $(x_0, y_0)$ is an equilibrium point. If the linearization at $(x_0, y_0)$ has a generic equilibrium point at the origin, then the equilibrium point at $(x_0, y_0)$ is of the same type.
Stability for $D \geq 1$

**Theorem:** Suppose that $y_0$ is an equilibrium point for $y' = f(y)$. Let $J$ be the Jacobian of $f$ at $y_0$.

1. Suppose that the real part of every eigenvalue of $J$ is negative. Then $y_0$ is an asymptotically stable equilibrium point.

2. Suppose that $J$ has at least one eigenvalue with positive real part. Then $y_0$ is an unstable equilibrium point.

Global Qualitative Analysis

- What happens to all solutions as $t \to \infty$?
- The (forward) limit set of the solution $y(t)$ that starts at $y_0$ is the set of all limit points of the solution curve. It is denoted by $\omega(y_0)$.
  - $x \in \omega(y_0)$ if there is a sequence $t_k \to \infty$ such that $y(t_k) \to x$.
- What is $\omega(y_0)$ for all $y_0$?
  - What is the limit set for all solutions?
  - In dimension 1, all limit sets are equilibrium points.

Limit Sets in Dimension 2

**Theorem:** If $S$ is a nonempty limit set of a solution of a planar system defined in a set $U \subset \mathbb{R}^2$, then $S$ is one of the following:

- An equilibrium point.
- A closed solution curve.
  - A closed solution curve is its own limit set.
  - The closed solution curve could be a limit cycle.
- A directed planar graph with vertices that are equilibrium points, and edges which are solution curves.

These are called the Poincaré-Bendixson alternatives.

- In dimension 3 the answer is unknown.
Poincaré-Bendixson Theorem

Theorem: Suppose that $R$ is a closed and bounded planar region that is positively invariant for a planar system. If $R$ contains no equilibrium points, then there is a closed solution curve in $R$.

- The theorem is also true if the set $R$ is negatively invariant.
- The closed solution curve might be a limit cycle.

Invariant Sets

Definition: A set $S$ is (positively) invariant for the system $y' = f(y)$ if $y(0) = y_0 \in S$ implies that $y(t) \in S$ for all $t \geq 0$.

- Examples include equilibrium points, and any solution curve.
- In dimension 2, invariant sets can frequently be found using:
  - nullclines,
  - polar coordinates.

Solving Separable Equations

\[ \frac{dy}{dt} = g(y)h(t) \]

The three step solution process:

1. Separate the variables. \[ \frac{dy}{g(y)} = h(t) \, dt \] if $g(y) \neq 0$.
2. Integrate both sides. \[ \int \frac{dy}{g(y)} = \int h(t) \, dt \]
3. Solve for $y(t)$.  

8 John C. Polking
Solving the Linear Equation $x' = a(t)x + f(t)$

Four step process:
1. Rewrite as $x' - ax = f$.
2. Multiply by the integrating factor
   
   $u(t) = e^{-\int a(t) \, dt}$.

   Equation becomes $[ux]' = u x' - ax = uf$.
3. Integrate: $u(t)x(t) = \int u(t)f(t) \, dt + C$.
4. Solve for $x(t)$.

Keys to Computing $e^{tA}v$

- If $v$ is a vector, we have
  
  $e^{tA}v = v + tAv + \frac{t^2}{2!} A^2v + \cdots$

- Truncation:
  - If $Av = 0$, then $e^{tA}v = v$.
  - If $A^k v = 0$, then $e^{tA}v = v + tAv + \frac{t^2}{2!} A^2v + \cdots + \frac{t^{k-1}}{(k-1)!} A^{k-1}v$.

- Law of Exponents: If $AB = BA$, then $e^{A+B} = e^A e^B$.
- Implies that $e^{tA} = e^{t\mu} e^{t[A-\lambda]}$.

Proposition

Proposition: Suppose that $A$ is an $n \times n$ matrix, $\lambda$ is a number, and $v$ is a vector.

1. If $[A - \lambda]v = 0$, then $e^{tA}v = e^{t\lambda}v$.
2. If $[A - \lambda]^2 v = 0$, then $e^{tA}v = e^{t\lambda}(v + t[A - \lambda]v)$.
3. If $[A - \lambda]^3 v = 0$, then
   
   $e^{tA}v = e^{t\lambda} \left( v + t[A - \lambda]v + \frac{t^2}{2!} [A - \lambda]^2v + \cdots \right.
   
   \left. + \frac{t^{k-1}}{(k-1)!} [A - \lambda]^{k-1}v \right)$.  

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Eigenvalues and Eigenvectors

- $\lambda$ is an eigenvalue of $A$ if there is a nonzero vector $v$ such that $Av = \lambda v$. If $\lambda$ is an eigenvalue of $A$, then any vector $v$ such that $Av = \lambda v$ is called an eigenvector associated with $\lambda$.

- $\lambda$ is an eigenvalue of $A$ if $p(\lambda) = \det(A - \lambda I) = 0$. $p(\lambda)$ is the characteristic polynomial of $A$.

- If $\lambda$ is an eigenvalue of $A$ and $(A - \lambda)^p v = 0$ for some integer $p \geq 1$, then $v$ is called a generalized eigenvector associated with $\lambda$. We can compute $e^{\lambda t} v$ for any generalized eigenvector.

Multiplicities

If $A$ is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$.

- The characteristic polynomial has the form $p(\lambda) = (\lambda - \lambda_1)^{q_1} (\lambda - \lambda_2)^{q_2} \cdots (\lambda - \lambda_k)^{q_k}$.

- The algebraic multiplicity of $\lambda_j$ is $q_j$.

- The geometric multiplicity of $\lambda_j$ is $d_j$, the dimension of the eigenspace of $\lambda_j$.

- $1 \leq d_j \leq q_j$.

- There is an integer $k_j \leq q_j$ for which null$(A - \lambda_j I)^{k_j}$ has dimension $q_j$.

Replacing Complex Solutions with Real Solutions

- If $A$ has complex eigenvalues, the fundamental set of solutions contains complex valued solutions.

- Complex solutions occur in complex conjugate pairs $z(t) = x(t) + iy(t)$ and $\overline{z(t)} = x(t) - iy(t)$.

- Replace $z(t)$ and $\overline{z(t)}$ with the real solutions $x(t) = \text{Re}(z(t))$ and $y(t) = \text{Im}(z(t))$. 