Math 211
Lecture #11
Financial Models

September 21, 2003

Compound Interest

Put some money into an account that returns a percentage each year, compounded continuously. How fast will it grow?

- \( P(t) \) is the principal balance measured in $1000.
- “Some money” is \( P(0) = P_0 \).
- “Returns a percentage” is \( r \)%/year.
- “Some time later” is measured in years.
- “Compounded continuously” means \( P' = rP \).
- The solution is \( P(t) = P_0 e^{rt} \).
- The principal grows exponentially.
- If \( r = 8\% \), then \( P(20) = P_0 e^{0.08 \times 20} = 4.953 P_0 \)
- \( P(40) = 24.5325 P_0 \).

Returns on Investments

What rates of return can we expect?

- Checking accounts — 0 – 3%.
- Money market accounts — 1/4 – 3%.
- Certificates of deposit (3 years) 3 – 4 %.
- Industrial bonds — 5.3% (average from 1926 – 2001).
- Stocks — 10.7% (average from 1926 – 2001).
Retirement Account

- Set up a retirement account by investing an initial amount. In addition, deposit a fixed amount each year until you retire. Assume it returns a percentage each year, compounded continuously. How much is there some time later?
- "A fixed amount each year" is $D$, measured in $1,000 each year. We assume this is invested continuously.
- The model is $P' = rP + D$.
- The solution is $P(t) = P_0e^{rt} + \frac{D}{r}[e^{rt} - 1]$.

Example of a Retirement Account

- Suppose you start with an investment of $1,000 at the age of 25, and invest $100 each month until you retire at 65. The account returns 8% per year. How much is in the retirement account when you retire?
  - $P_0 = 1000, D = 100 \times 12 = 1200, r = 8\% = 0.08$.
  - At 65 the principal is $377,521.
  - Is this enough to retire on?

Retirement Planning

- If you need a certain income after you retire, how much must you have in your retirement account when you retire?
  - "Certain income" is $I$ (in $1000/year) withdrawn from the account.
  - "How much" is the amount $P_0$ in the account at retirement.
  - The account still grows due to its return at $r\%$/year.
  - The model is $P' = rP - I$, $P(0) = P_0$.
  - The solution is $P(t) = P_0e^{rt} - \frac{I}{r}[e^{rt} - 1]$.
  - We are given $I$, $r$, & $P(t_d)$. We need to compute $P_0$. 
Retirement Planning – Example 1

• If you will need an income of $75,000 for 30 years after retirement and your account returns 6%, your account balance at retirement should be $1,043,000.

• How are you going to save over a million dollars?

Retirement Planning (second try)

• Instead of investing a fixed amount each month, it would be more realistic to invest a percentage of your salary. What should this percentage be in order to accumulate an adequate investment balance? Include the effect of inflation.

• You starting salary is $S_0$. Assume it will increase at $s$% per year.
  • Then $S' = sS$, or $S(t) = S_0e^{st}$.

• The model for the growth of the retirement account is $P' = rP + \lambda S_0 e^{st}$ with $P(0) = P_0$.

• The solution is $P(t) = P_0 e^{rt} + \frac{\lambda S_0}{r - s} (e^{rt} - e^{st})$.

Retirement Planning – Example 2

• Assume
  • $P_0 = $1,000 and $r = 8\%$
  • $S_0 = $35,000 and $s = 4\%$
    • Notice that $S(40) = $173,356.
  • Need a retirement income of $150,000.
    • Aim for a balance at retirement of $2,000,000.
  • Requires $\lambda = 11.53\%$.  
Other Strategies

- Delayed gratification. Deposit a percentage of your salary that starts at $\lambda\%$, and decays linearly to 0 over 40 years.

$$P' = rP + \lambda(1 - t/40)S_0 e^{rt}$$

- Immediate gratification. Deposit a percentage of your salary that starts at 0 and grows linearly over 40 years to $\lambda\%$.

$$P' = rP + \frac{\lambda}{40} S_0 e^{rt}$$