HW 11:

§ 9.6  6, 7, 20, 27, 28
§ 9.8  2
This matrix has a nullspace of dimension 1, so the eigenvalue \( \lambda = -1 + i \) has geometric multiplicity 1. Our computer tells us that \( \mathbf{w}_1 = (1 + i, 0, 2 + 2i, -1)^T \) is an eigenvector. The corresponding complex-valued solution is

\[
\mathbf{z}_1(t) = e^{tA} \mathbf{w}_1 = e^{(-1+i)t} \mathbf{w}_1 = e^{(-1+i)t} \begin{pmatrix} 1+i \\ 0 \\ 2+2i \\ -1 \end{pmatrix}.
\]

To find another complex-valued solution, we compute

\[
(A - (-1 + i)I)^2 = \begin{pmatrix} 2 - 14i & 3 - 12i & -2 + 6i & -4i \\ 8i & -2 + 6i & -4i & 0 \\ 8 - 16i & 6 - 14i & -6 + 6i & -8i \\ -2 - 2i & 1 + 2i & -2 + 2i \end{pmatrix}.
\]

We look for a vector \( \mathbf{w}_2 \) in this nullspace that is not an eigenvector and therefore not a multiple of \( \mathbf{w}_1 \). One choice is \( \mathbf{w}_2 = (1 + 2i, -2 - 2i, 0, 2)^T \). Since

\[
(A - (-1 + i)I)^2 \mathbf{w}_2 = 0,
\]

the corresponding solution is

\[
\mathbf{z}_2(t) = e^{tA} \mathbf{w}_2 = e^{(-1+i)t} \left( \mathbf{w}_2 + t(A - (-1 + i)I)\mathbf{w}_2 \right) = e^{(-1+i)t} \begin{pmatrix} (1+t) + i(2t) \\ -2 - 2i \\ 2t + 2it \\ 2 - t \end{pmatrix}.
\]

Corresponding to the complex conjugate eigenvalue \(-1 - i\), we have the conjugate generalized eigenvectors \( \overline{\mathbf{w}}_1 \) and \( \overline{\mathbf{w}}_2 \) and the corresponding conjugate solutions \( \overline{\mathbf{z}}_1 \) and \( \overline{\mathbf{z}}_2 \). These are the required four solutions. To find real solutions, we use the real and imaginary parts of the complex solutions. If \( \mathbf{z}_1 = x_1 + iy_1 \) and \( \mathbf{z}_2 = x_2 + iy_2 \), then \( x_1, x_2, y_1, y_2 \) are the four needed real solutions.

**EXERCISES**

Use Definition 6.5 to calculate \( e^A \) for the matrices in Exercises 1–4.

1. \( A = \begin{pmatrix} -2 & 1 \\ 1 & 2 \end{pmatrix} \)
2. \( A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \)
3. \( A = \begin{pmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \)
4. \( A = \begin{pmatrix} -2 & 1 & -3 \\ -1 & 1 & -1 \end{pmatrix} \)

5. Suppose that the matrix \( A \) satisfies \( A^2 = \alpha A \), where \( \alpha \neq 0 \).
   (a) Use Definition 6.5 to show that
   \[
   e^A = I + \frac{e^{\alpha t} - 1}{\alpha} A.
   \]

6. There are many important series in mathematics, such as the exponential series. For example,

\[
\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \quad \text{and} \quad \sin t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots.
\]

Use these infinite series together with Definition 6.5 to show that

\[
e^{(0 - 1)} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.
\]

(b) Use part (a) to compute \( e^A \) for

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}.
\]

7. Use the result of Exercise 6 to find \( e^A \).

8. If

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

find \( e^A \).

9. Let

\[
A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}
\]

(a) Show how you would do the exercise.
(b) Evaluate the result you obtained.
(c) Use Definition 6.5 to get the result for your exercise.

10. If \( A = P L \), find \( e^A \) using the result of Exercise 11.

11. \( A = \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix} \)

Use the results of Exercise 10 to calculate \( e^A \).

12. \( A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \)

There are many important matrices in mathematics, such as the exponential series. For example,

\[
\cos t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{(2k)!} = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \cdots \quad \text{and} \quad \sin t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!} = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \cdots.
\]

Use these infinite series together with Definition 6.5 to show that

\[
e^{(0 - 1)} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.
\]

13. Let \( A \) be a nilpotent matrix so that

\[
(A - \lambda I)^k = 0 \quad \text{for some integer } k > 0.
\]

In Exercises 14–16 find \( e^A \) for each matrix, using the result of Exercise 13.

14. \( A = \begin{pmatrix} -2 & -1 \\ -1 & 0 \end{pmatrix} \)

15. \( A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \)

16. \( A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \)

Each of the matrices has eigenvalue \( \lambda \). In each case,

\[
(A - \lambda I)^k = 0.
\]

\[
e^A = e^{\lambda I} = I + \frac{e^{\lambda t} - 1}{\lambda} A.
\]

Use Definition 6.5 to compute \( e^A \).

17. \( A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \)

18. \( A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \)
7. Use the result of Exercise 6 to show that if
\[ A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \]
then
\[ e^{At} = e^{at} \begin{pmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{pmatrix}. \]

**Hint:** If \( A = at + b \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) \).

8. If
\[ A = \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \]
find \( e^{At} \). **Hint:** See the hint for Exercise 7.

9. Let
\[ A = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \]

(a) Show that \( AB \neq BA \).

(b) Evaluate \( e^{A+B} \). **Hint:** This is a simple computation if you use Exercise 7.

(c) Use Definition 6.5 to evaluate \( e^A \) and \( e^B \). Use these results to compute \( e^A e^B \) and compare this with the result found in part (b). What have you learned from this exercise?

10. If \( A = P D P^{-1} \), prove that \( e^{At} = P e^{tD} P^{-1} \).

Use the results of Exercise 53 of Section 9.1 and Exercise 10 to calculate \( e^{At} \) for each matrix in Exercises 11–12.

11. \( A = \begin{pmatrix} -2 & 6 \\ 0 & -1 \end{pmatrix} \)

12. \( A = \begin{pmatrix} -2 & 0 \\ -3 & -3 \end{pmatrix} \)

13. Let \( A \) be a \( 2 \times 2 \) matrix with a single eigenvalue \( \lambda \) of algebraic multiplicity 2 and geometric multiplicity 1. Prove that
\[ e^{At} = e^{\lambda t} [I + (A - \lambda I)t]. \]

Exercises 14–17, each matrix has an eigenvalue of algebraic multiplicity 2 but geometric multiplicity 1. Use the technique Exercise 13 to compute \( e^{At} \).

14. \( A = \begin{pmatrix} -2 & 1 \\ -1 & 0 \end{pmatrix} \)

15. \( A = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix} \)

16. \( A = \begin{pmatrix} 0 & -1 \\ 1 & -2 \end{pmatrix} \)

17. \( A = \begin{pmatrix} -3 & -1 \\ 4 & 1 \end{pmatrix} \)

18. For each of the matrices in Exercises 18–25, use the fact that \( e^A = e^{\lambda t} \left( I + t(A - \lambda I) + \frac{t^2}{2!}(A - \lambda I)^2 + \cdots \right) \) to compute \( e^{At} \).

19. \( A = \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \)

20. \( A = \begin{pmatrix} -2 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & -4 \end{pmatrix} \)

21. \( A = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & 0 \\ -1 & 1 & -2 \end{pmatrix} \)

22. \( A = \begin{pmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)

23. \( A = \begin{pmatrix} -5 & 0 & -1 \\ -4 & 0 & 1 \\ 4 & -4 & -5 \end{pmatrix} \)

24. \( A = \begin{pmatrix} 0 & 4 & 5 \\ -1 & -1 & -2 \end{pmatrix} \)

25. \( A = \begin{pmatrix} 1 & 0 & 0 \\ -9 & 4 & 1 \\ 13 & -3 & -5 \end{pmatrix} \)

26. \( A = \begin{pmatrix} -2 & 1 & -1 \\ 1 & -3 & 0 \\ 3 & -5 & 0 \end{pmatrix} \)

27. \( A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & -2 \\ 0 & 0 & 2 \end{pmatrix} \)

28. \( A = \begin{pmatrix} 0 & 1 & 0 \\ -4 & 4 & 0 \\ 0 & -2 & 0 \end{pmatrix} \)

29. \( A = \begin{pmatrix} -1 & 0 & 0 \\ 2 & -5 & -1 \\ 0 & 4 & -1 \end{pmatrix} \)

30. \( A = \begin{pmatrix} 11 & -12 & 42 \\ -12 & 39 & -4 \\ -24 & 81 & -57 \end{pmatrix} \)

31. \( A = \begin{pmatrix} 18 & -7 & 24 \\ 15 & -8 & 20 \\ 0 & 0 & -1 \end{pmatrix} \)

Do the following for each of the matrices in Exercises 26–33. Exercises 26–29 can be done by hand, but you should use a computer for the rest.

(i) Find the eigenvalues.

(ii) For each eigenvalue, find the algebraic and the geometric multiplicities.

(iii) For each eigenvalue \( \lambda \), find the smallest integer \( k \) such that the dimension of the nullspace of \( (A - \lambda I)^k \) is equal to the algebraic multiplicity.

(iv) For each eigenvalue \( \lambda \), find \( q \) linearly independent generalized eigenvectors, where \( q \) is the algebraic multiplicity of \( \lambda \).

(v) Verify that the collection of the generalized eigenvectors you find in part (iv) for all of the eigenvalues is linearly independent.

(vi) Find a fundamental set of solutions for the system \( y' = Ay \).
Now suppose that \( \lambda = \alpha + i\beta \) is a complex root of multiplicity \( q \). Then \( \alpha - i\beta \) is also a complex root of multiplicity \( q \). By the same reasoning we used for the real case, we get complex conjugate pairs of solutions

\[
z(t) = t^k e^{\lambda t} \quad \text{and} \quad \overline{z}(t) = t^k e^{\overline{\lambda} t}, \quad \text{where} \quad k = 0, 1, \ldots, q - 1.
\]

To find the real solutions, we take the real and imaginary parts,

\[
x(t) = t^k e^{\alpha t} \cos \beta t \quad \text{and} \quad y(t) = t^k e^{\alpha t} \sin \beta t, \quad \text{where} \quad k = 0, 1, \ldots, q - 1.
\]

Let's summarize the facts about the solutions associated with the complex root \( \lambda = \alpha + i\beta \).

**Theorem 8.36** If \( \lambda = \alpha + i\beta \) is a complex root of the characteristic polynomial with multiplicity \( q \), then so is \( \lambda = \alpha - i\beta \). In addition,

\[
x_1(t) = e^{\alpha t} \cos \beta t, \quad x_2(t) = t e^{\alpha t} \cos \beta t, \quad \ldots,
\]

and \( x_q(t) = t^{q-1} e^{\alpha t} \cos \beta t \)

\[
y_1(t) = e^{\alpha t} \sin \beta t, \quad y_2(t) = t e^{\alpha t} \sin \beta t, \quad \ldots,
\]

and \( y_q(t) = t^{q-1} e^{\alpha t} \sin \beta t \)

are \( 2q \) linearly independent solutions.

**Example 8.37** Find a fundamental set of solutions to

\[
y'''' + 4y''' + 14y'' + 20y' + 25y = 0.
\]

The characteristic polynomial is

\[
\lambda^4 + 4\lambda^3 + 14\lambda^2 + 20\lambda + 25 = (\lambda^2 + 2\lambda + 5)^2.
\]

Consequently, we have roots \(-1 \pm 2i\), each of multiplicity 2. Thus, we have solutions

\[
y_1(t) = e^{-t} \cos 2t, \quad y_2(t) = e^{-t} \sin 2t, \quad y_3(t) = te^{-t} \cos 2t,
\]

and \( y_4(t) = te^{-t} \sin 2t \).

**Exercises (59.8)**

1. The function \( y(t) \) is a solution of the homogeneous equation \( y'' - 2y' - 3y = 0 \) if and only if

\[
x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}
\]

is a solution of

\[
x' = \begin{pmatrix} 0 & 1 \\ 3 & 2 \end{pmatrix} x.
\]

(a) Use direct substitution to show that

\[
x_1(t) = \begin{pmatrix} e^{3t} \\ 3e^{3t} \end{pmatrix} \quad \text{and} \quad x_2(t) = \begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}
\]

are solutions of system (1). Show that \( x_1(t) \) and \( x_2(t) \) are linearly independent.

(b) Use direct substitution to show that the first component of the general solution \( x(t) = C_1 x_1(t) + C_2 x_2(t) \) is a solution of \( y'' - 2y' - 3y = 0 \).

2. The function \( y(t) \) is a solution of the homogeneous equation \( y''' + 4y = 0 \) if and only if

\[
x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} y(t) \\ y'(t) \end{pmatrix}
\]

is a solution of

\[
x' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix} x.
\]

(a) Use direct substitution to show that

\[
x_1(t) = \begin{pmatrix} \sin 2t \\ 2 \cos 2t \end{pmatrix} \quad \text{and} \quad x_2(t) = \begin{pmatrix} \cos 2t \\ -2 \sin 2t \end{pmatrix}
\]

are solutions of system (2). Show that \( x_1(t) \) and \( x_2(t) \) are linearly independent.

(b) Use direct substitution to show that the first component of the general solution \( x(t) = C_1 x_1(t) + C_2 x_2(t) \) is a solution of \( y''' + 4y = 0 \).

Each equation possesses a fundamental set of solutions.

14. \( y'''' + 16y'' + 64y = 0 \)
16. \( y^{(4)} + 8y'' + 16y = 0 \)
18. \( y^{(4)} - 4y'' + 4y = 0 \)
20. \( y^{(5)} + 10y'' + 25y = 0 \)
are solutions of system (2). Show that \( x_1(t) \) and \( x_2(t) \) are linearly independent.

(b) Use direct substitution to show that the first component of the general solution \( x(t) = C_1x_1(t) + C_2x_2(t) \) is a solution of \( y'' + 4y = 0 \).

Use Definition 8.15 and the technique of Example 8.16 to show that each set of functions in Exercises 3–6 is linearly independent.

3. \( y_1(t) = e^t \) and \( y_2(t) = e^{2t} \)
4. \( y_1(t) = e^t \cos t \) and \( y_2(t) = e^t \sin t \)
5. \( y_1(t) = \cos t \), \( y_2(t) = \sin t \), and \( y_3(t) = e^t \)
6. \( y_1(t) = e^t \), \( y_2(t) = te^t \), and \( y_3(t) = t^2e^t \)

In Exercises 7–12, use Proposition 8.19 and the technique of Example 8.20 to show that the given solutions are linearly independent and form a fundamental set of solutions of the given equation.

7. The equation \( y'' + 9y = 0 \) has solutions \( y_1(t) = \cos 3t \) and \( y_2(t) = \sin 3t \).
8. The equation \( y'' + 9y' - 10y = 0 \) has solutions \( y_1(t) = e^{-10t} \) and \( y_2(t) = e^{t} \).
9. The equation \( y'' - 4y' + 4y = 0 \) has solutions \( y_1(t) = e^{2t} \) and \( y_2(t) = te^{2t} \).
10. The equation \( y''' - 3y'' + 9y' - 27y = 0 \) has solutions \( y_1(t) = \cos 3t \), \( y_2(t) = \sin 3t \), and \( y_3(t) = e^{3t} \).
11. The equation \( y''' - 3y'' - 3y' + y = 0 \) has solutions \( y_1(t) = e^t \), \( y_2(t) = te^t \), and \( y_3(t) = t^2e^t \).
12. The equation \( y^{(6)} - 13y'' + 36y = 0 \) has solutions \( y_1(t) = \cos 3t \), \( y_2(t) = \sin 3t \), \( y_3(t) = \cos 2t \), and \( y_4(t) = \sin 2t \).

13. Consider the equation

\[ y''' + ay'' + by' + cy = 0. \] (8.38)

(a) If \( e^{x} \) is a solution of equation (8.38), provide details showing that \( x^3 + ax^2 + bx + c = 0 \).

(b) Write the third-order equation (8.38) as a system of first-order equations, placing your answer in the form \( \mathbf{x}' = A\mathbf{x} \). Calculate the characteristic polynomial of system \( \mathbf{x}' = A\mathbf{x} \) and compare it with the characteristic polynomial of the equation in (8.38).

Each equation in Exercises 14–21 has a characteristic equation possessing distinct real roots. Find the general solution of each equation.

14. \( y''' - 2y'' - y' + 2y = 0 \)
15. \( y''' - 3y' = 4y' - 12y \)
16. \( y^{(6)} - 5y'' + 4y = 0 \)
17. \( y^{(6)} + 36y = 13y'' \)
18. \( y'' + 2y' - 5y' - 6y = 0 \)
19. \( y''' + 30y = 4y'' + 11y' \)
20. \( y^{(5)} - 4y'' - 13y' + 52y'' - 36y' - 144y = 0 \)

Each equation in Exercises 22–27 has a characteristic equation possessing real roots of various multiplicities. Find the general solution of each equation.

22. \( y''' - 3y' + 2y = 0 \)
23. \( y''' + y'' = 8y' + 12y \)
24. \( y''' + 6y'' + 12y' + 8y = 0 \)
25. \( y''' + 3y'' + 3y' + y = 0 \)
26. \( y^{(6)} + 3y'' - 6y' - 10y'' + 21y' - 9y = 0 \)
27. \( y^{(6)} - 3y'' - 6y' - 11y' + 3y = 0 \)

Each equation in Exercises 28–33 has a characteristic equation possessing some complex zeros, some of which are repeated. Find the general solution of each equation.

28. \( y''' - y'' + 4y' - 4y = 0 \)
29. \( y''' + 2y'' = y' \)
30. \( y^{(6)} + 17y'' + 16y = 0 \)
31. \( y^{(6)} + y = -2y'' \)
32. \( y^{(6)} - 9y' + 34y'' - 66y' + 65y'' - 25y = 0 \)
33. \( y^{(6)} + 3y'' + 3y' + y = 0 \)

Find the solution of each initial-value problem presented in Exercises 34–43.

34. \( y''' - 2y' - 3y = 0 \), with \( y(0) = 4 \) and \( y'(0) = 0 \)
35. \( y''' + 2y' + 5y = 0 \), with \( y(0) = 2 \) and \( y'(0) = 0 \)
36. \( y''' + 4y' + 4y = 0 \), with \( y(0) = 2 \) and \( y'(0) = -1 \)
37. \( y''' - 2y' + y = 0 \), with \( y(0) = 1 \) and \( y'(0) = 0 \)
38. \( y''' - 4y'' - 7y' + 10y = 0 \), with \( y(0) = 1 \), \( y'(0) = 0 \), and \( y''(0) = 1 \)
39. \( y''' - 7y'' + 11y' - 5y = 0 \), with \( y(0) = -1 \), \( y'(0) = 1 \), and \( y''(0) = 0 \)
40. \( y''' - 2y' + 4y = 0 \), with \( y(0) = 1 \), \( y'(0) = -1 \), and \( y''(0) = 0 \)
41. \( y''' - 6y'' + 12y' - 8y = 0 \), with \( y(0) = -2 \), \( y'(0) = 0 \), and \( y''(0) = 2 \)
42. \( y''' - 3y'' + 52y = 0 \), with \( y(0) = 0 \), \( y'(0) = -1 \), and \( y''(0) = 2 \)
43. \( y^{(6)} + 8y'' + 16y = 0 \), with \( y(0) = 0 \), \( y'(0) = -1 \), \( y''(0) = 2 \), and \( y'''(0) = 0 \)

In Exercises 44–46, we will verify some of the steps leading to the proof of Theorem 8.33.

44. Verify (8.30).
45. Verify (8.31). See the definition of a subspace in Definition 5.5 in Section 7.5.
46. Verify (8.32). Hint: Suppose that there are constants \( c_1, \ldots, c_k \) such that \( c_1a_1 + \cdots + c_ka_k = 0 \). Show that \( c_1y_1(t) + \cdots + c_ky_k(t) = 0 \). Conclude that all of the constants must be equal to zero.