ODE. Lecture I.

Notations:
- Defn
- Thm
- Ex
- ⇒
- Rmk
- Pf
- HW
- \[ \text{Definition} \]
- \[ \text{Theorem} \]
- \[ \text{Example} \]
- \[ \text{Remark} \]
- \[ \text{Proof} \]
- \[ \text{homework} \]

Homework will be assigned everyday after class and collected before the class on each M, W, F.
Questions are encouraged to ask during the class.
After every lecture, I will try my best to upload lectures notes for that day on the webpage.
The class on everyday is composed of 3 lectures: 1:00-1:50, 2:00-2:50, 3:00-3:50.
The rest part, please read syllabus on the webpage.

As concerned with next Monday, I heard there won't be class on that day. But I will make corresponding adjustment about homework and exams when I make sure of more.
Defn. An ordinary equation is an equation involving an unknown function of a single variable together with one or more of its derivatives.

Ex. \( xy' + 4y = x \) \( \text{(1)} \)

\( y' = y^2 - t \) \( \text{(2)} \)

\( ty' = y \) \( \text{(3)} \)

\( y'' + t^2y = \cos t \) \( \text{(4)} \)

Rmk: \( \text{(1), (2), (3)} \) are first-order equations.

\( \text{(4)} \) is second-order equations.

hence \( ty'' + ty' + y' = \sin t \)

is third-order equations.

To solve things more general, we place differential equations into \[ \text{normal form} \]

Defn. A first-order differential equation of the form \( y' = f(t, y) \)

is said to be in normal form.

Similarly, an equation of order \( n \) has the form \( y^{(n)} = f(t, y', y'', \ldots, y^{(n-1)}) \) is said to be in normal form.
Defn. A direction field is an association of little line elts to every pt on the plane.

Rmk: We always just draw little line elts on some pts to represent it.

Like: Can first consider a lattice.

Draw little line elts at these pts.

1. For \( y' = f(x, y) \), how to get a direction field of \( y' = f(x, y) \).
   At pt \((x, y)\), draw little line elt with slope \( f(x, y) \).

Ex. \( y' = y = f(x, y) \)

\[ \begin{array}{ll}
\text{On } (x, y) &= (0, 0), \ f(x, y) = y = 0 . \\
\text{On } (x, y) &= (x, 0), \ f(x, y) = y = 0 . \\
\text{On } (x, y) &= (x, 1), \ f(x, y) = y = 1 . \\
\text{On } (x, y) &= (x, 2), \ f(x, y) = y = 2 . \\
\text{On } (x, y) &= (x, -1), \ f(x, y) = y = -1 . \\
\text{On } (x, y) &= (x, -2), \ f(x, y) = y = -2 . 
\end{array} \]
Defn. An integral curve of a direction field is a curve which has the direction of direction field as tangent vectors at every pt of this curve.

Ex. as above.

Remark: the integral curve is the graph of solution $y = y(x)$ to $y' = f(x, y)$.

so: solving equation of $y' = f(x, y) \iff$ draw integral curve of direction field of $y' = f(x, y)$. 
Now we first consider:

**First Order ODE's**

\[ y' = f(x, y) \]

\[ \leftarrow \text{written in normal form} \]

Ex. \[ y' = \frac{x}{y} \]
\[ \quad \leftarrow \text{solvable, separable equation} \]
\[ y' = x - y^2 \]
\[ \quad \leftarrow \text{unsolvable, (namely, no elementary linear equation)} \]
\[ y' = y - x^2 \]
\[ \quad \leftarrow \text{solvable} \]

**Geometric View of ODE's**

Analytic:

\[ y' = f(x, y) \]
\[ \quad \leftarrow \text{(1) Direction field} \]

a solution \( y(x) \)
\[ \leftarrow \text{(2) Integral curve} \]

(It can have many solutions)

Let me explain what's "Direction field" and "Integral curve".

(See next page).
i.e. \( y(x) \) is solution to \( y' = f(x, y) \) \( \iff \) graph of \( y(x) \) is an integral curve of direction field of \( y' = f(x, y) \).

What the left side mean?

\( y'(x) = f(x, y(x)) \).

\[ \begin{array}{c}
\text{Same!}
\end{array} \]

So we see they mean same thing!

What the right side mean?

slope of \( y(x) \) = slope of direction field at \( (x, y(x)) \),

exactly \( f(x, y(x)) \).

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**Draw Direction Field:**

- **Computer Method:**
  1. Pick \((x, y)\).
  2. \( f(x, y) = \) final calculation.
  3. Draw a little line elt has slope \( f(x, y) \) at \((x, y)\).

- **Human Method:**
  1. Pick Slope \( C \) (constant \( C \)).
  2. \( f(x, y) = C \). Plot the equation. Find \( x, y \) satisfy this equation which is a curve of slope \( C \).
  3. make the curve dotted, and draw little line elt's based on pts of dotted curve.

Ex. \( C = 0 \).
Ex. \( y' = -\frac{x}{y} \).

So \( \frac{x}{y} = C \). \[ y = -\frac{1}{C}x \]

\[ C = 0 \quad \text{"C = -1"} \]
\[ C = 1 \quad \text{"C = \infty"} \]

integral curve.

are circles: \( x^2 + y^2 = C \).

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**Initial Value Problem.**

Defn. A first order differential equation together with an initial condition,

\[ y' = f(x, y), \quad y(t_0) = y_0, \]

is called an initial value problem.

Ex. \( y' = y^2 - t, \quad y(4) = 0 \).

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**Interval of Existence.**

Defn. The interval of existence of a solution to a differential equation is defined to be the largest interval over which the solution can be defined and remain a solution.
Solutions to Separable Equations.

Defn. Separable differential Equation.

Equations of form like \( \frac{dy}{dt} = g(t)f(y) \).

are called separable differential equations.

We can solve separable equation using the following 3 steps:

1. Separate the variables: \( \frac{dy}{f(y)} = g(t)dt \).

2. Integrate both sides: \( \int \frac{dy}{f(y)} = \int g(t)dt \).

3. Solve for the equation \( y(t) \), if possible.

Ex. \( y' = \frac{e^x}{(1+y)} \)

Separable? \( y' = \cos(xy) \)

\( y' = xy + y \)

\( y' = e^{y/x} \)

\( y' = \frac{e^y}{y} \)
Ex : \( y' = ty^2 \).

Step 1 : Write the equation using \( \frac{dy}{dt} \) instead of \( y' \),
so \( \frac{dy}{dt} = ty^2 \).

Step 2 : \( \frac{dy}{y^2} = t \, dt \).

Integrate it.

\[
\int \frac{1}{y^2} \, dy = \int t \, dt .
\]

\[\Rightarrow -\frac{1}{y} = \frac{1}{2}t^2 + C.\]

Step 3 : Solve it, get the solution is

\[
y(t) = -\frac{1}{\frac{1}{2}t^2 + C} = \frac{-2}{t^2 + 2C}.
\]
Ex. \( x' = \frac{2t \cdot x}{1 + x} \), \( x(0) = 1 \).

**Step 1:** \( \frac{dx}{dt} = \frac{2t \cdot x}{1 + x} \).

\( \Rightarrow \) \( \frac{1 + x}{x} \) \( dx = 2t \) \( dt \)

\( \Rightarrow \) \( (1 + \frac{1}{x}) dx = 2t \) \( dt \)

**Step 2.** Integrate it, \( \int (1 + \frac{1}{x}) dx = \int 2t \) \( dt \).

\( \Rightarrow \) \( x + \ln|x| = t^2 + C \)

**Step 3.** Solve for \( C \).

Plug in \( x(0) = 1 \), i.e. \( t = 0 \), \( x = 1 \).

\( \Rightarrow \) \( 1 + \ln 1 = 0 + C \)

so \( 1 = C \).

hence \( x + \ln|x| = t^2 + 1 \).

**Discussion:** The function \( \ln|x| \) is not defined at \( x = 0 \), so our solution can never be equal to 0.

Since our initial condition is positive, and a solution must be continuous, our solution \( x(t) \) must be positive for all \( t \). So \( |x| = x \) and our solution is given implicitly by \( x + \ln x = t^2 + 1 \). (\( \star \))

This is as far as we can go. We cannot solve this equation. So just say the solution is defined implicitly by (\( \star \)).
Ex. \( y' = (\sin x)/y \quad y(\frac{\pi}{2}) = 1 \).

Step 1: \( \frac{dy}{dx} = \sin x/y \). 

\[ \Rightarrow y \, dy = \sin x \, dx \]

Step 2: Integrate it.

\[ \int y \, dy = \int \sin x \, dx \]

\[ \Rightarrow \frac{1}{2} y^2 = -\cos x + C \]

Step 3: Solve for \( C \).

Plug in \( y(\frac{\pi}{2}) = 1 \), i.e., \( -\frac{\pi}{2} \); \( y = 1 \).

\[ \Rightarrow \frac{1}{2} \cdot 1^2 = -\cos \frac{\pi}{2} + C \]

\[ = 0 + C \]

\[ \Rightarrow C = \frac{1}{2} \]

So \( \frac{1}{2} y^2 = -\cos x + \frac{1}{2} \)

hence \( y^2 = -2\cos x + 1 \).

\[ y(x) = \pm \sqrt{-2\cos x + 1} \]

Check: \( y(\frac{\pi}{2}) = 1 \).

so only \( y(x) = \sqrt{-2\cos x + 1} \) satisfies \( y(\frac{\pi}{2}) = 1 \).

hence \( y(x) = \sqrt{-2\cos x + 1} \) is the solution.
\[ y' = \frac{y}{x} \]

**Step 1:** Write the equation using \( \frac{dy}{dx} \) instead of \( y' \).

\[ \frac{dy}{dx} = \frac{y}{x} \]

**Step 2:** \( \frac{dy}{y} = \frac{dx}{x} \)

Integrate it:

\[ \int \frac{1}{y} \, dy = \int \frac{1}{x} \, dx \]

\[ \Rightarrow \ln y = \ln x + C. \]

**Step 3:** Solve it, get the solution:

\[ y(x) = e^{\ln x + C} = e^C x. \]

**Ex 3.** \( y' = (1 + y^2) e^x \).

**Step 1:** Write the equation using \( \frac{dy}{dx} \) instead of \( y' \).

\[ \frac{dy}{dx} = (1 + y^2) e^x \]

**Step 2:** \( \frac{dy}{1 + y^2} = e^x \, dx \)

Integrate it:

\[ \int \frac{1}{1 + y^2} \, dy = \int e^x \, dx \]

\[ \Rightarrow \arctan y = e^x + C. \]

**Step 3:** Solve it, get the solution:

\[ y(x) = \tan(e^x + C). \]
Solving Initial Value Problem. Need use definite integration.

Ex. \( y' = e^x/(1+y) \), \( y(0) = 1 \).

Step 1: \( \frac{dy}{dx} = e^x/(1+y) \)

\[ (1+y)dy = e^x \, dx \]

Step 2: Integrate it.

\[ \int (1+y) \, dy = \int e^x \, dx \]

\[ y + \frac{1}{2}y^2 = e^x + C \]

Step 3: Solve C.

Plug in \( y(0) = \frac{1}{2} \), i.e. \( x = 0, \ y = 1 \).

\[ 1 + \frac{1}{2} \cdot 1^2 = e^0 + C \]

\[ \Rightarrow C = \frac{1}{2} \]

Step 4: \( y + \frac{1}{2}y^2 = e^x + \frac{1}{2} \)

So \( 2y + y^2 - 2(e^x + \frac{1}{2}) = 0 \), i.e. \( y^2 + 2y - (2e^x + 1) = 0 \).

So \[ y = \frac{1}{2} \left[ -2 \pm \sqrt{4 + 4(2e^x + 1)} \right] \]

\[ = \frac{1}{2} \left[ -2 \pm 2\sqrt{2e^x + 2} \right] \]

\[ = -1 + \sqrt{2e^x + 2} \]

\[ \Box \]
Linear Equations.

Defn. A first-order linear equation is of the form

\[ x' = a(t) x + f(t). \]

If \( f(t) = 0 \), the equation is said to be homogeneous.

Otherwise it is inhomogeneous.

Ex. \[ x' = \sin t \cdot x. \]

\[ x' = e^{2t} x + \cos t. \] inhomogeneous.

\[ y' = x/y. \] nonlinear.

Solution of the homogeneous equation.

\[ x' = a(t) x. \]

\[ \Rightarrow \frac{dx}{dt} = a(t) x. \]

\[ \Rightarrow \frac{dx}{x} = a(t) \, dt \]

\[ \Rightarrow \int \frac{dx}{x} = \int a(t) \, dt \]

\[ \Rightarrow \ln |x| = \int a(t) \, dt + C. \]

Exponentiating, we get \[ |x| = e^{\int a(t) \, dt + C} = e^C \cdot e^{\int a(t) \, dt}. \]
Replace constant $e^C$ by $A$.

we get the general solution is

$$x(t) = A e^\int a(t) \, dt.$$ 

Ex. \quad $x' = \sin(t) \cdot x.$ \quad $\Rightarrow a(t) = \sin(t).$

get

$$x(t) = A e^{\int \sin(t) \, dt}$$

$$= A e^{-\cos(t)} + C$$

$$= A' e^{-\cos(t)}.$$

Solution of the inhomogeneous equation.

$$x' = a(t)x + f(t)$$

$$\Rightarrow x' - a(t)x = f(t)$$

then we don't know how to do it.

Now let's try an easier example.

If $a(t) = k$, i.e.

$$x' - kx = f(t).$$

we can try to solve it.

Recall: $(e^{-kt}x)' = e^{-kt}x' - e^{-kt} \cdot kx = e^{-kt}(x' - kx)$.

We can make use of this.

multiply $e^{-kt}$ on both sides of $x' - kx = f(t)$. 
We get 
\[ e^{-kt}(x' - kx) = e^{-kt}f(t), \]
\[ \Rightarrow (e^{-kt}x)' = e^{-kt}f(t) \]
\[ \Rightarrow \int (e^{-kt}x)' = \int e^{-kt}f(t) \]
\[ \Rightarrow e^{-kt}x = \int e^{-kt}f(t) \]
hence \[ x(t) = e^{kt} \int e^{-kt}f(t). \] [Good!]

So how about \[ x' - a(t)x = f(t). \]

Similarly, \[ (e^{-\int a(t)dt}x)' = e^{-\int a(t)dt}x' - e^{-\int a(t)dt}a(t)x \]
\[ = e^{-\int a(t)dt} (x' - a(t)x) \]

so multiply \[ e^{-\int a(t)dt} \] on both sides of \[ x' - a(t)x = f(t). \]
we get \[ e^{-\int a(t)dt} (x' - a(t)x) = e^{-\int a(t)dt}f(t). \]
\[ \Rightarrow (e^{-\int a(t)dt}x)' = e^{-\int a(t)dt}f(t) \]
\[ \Rightarrow \int (e^{-\int a(t)dt}x)' = \int e^{-\int a(t)dt}f(t) \]
\[ \Rightarrow e^{-\int a(t)dt}x = \int e^{-\int a(t)dt}f(t) \]
so \[ e^{-\int a(t)dt}x = \int e^{-\int a(t)dt}f(t) dt \]
\[ \Rightarrow x(t) = e^{\int a(t)dt} \int e^{-\int a(t)dt}f(t) dt \]
Summary of the method.

So we've found a general method of solving arbitrary linear equations.

For \[ x' = a(t)x + f(t) \],

it can be solved using following 4 steps.

1. Rewrite the equation as
   \[ x' - a(t)x = f(t) \].

2. Multiply by the integrating factor
   \[ u(t) = e^{-\int a(t)dt} \]
   so the equation becomes
   \[ (ux)' = u(x' - ax) = uf \]

3. Integrate this equation to obtain.
   \[ u(t)x(t) = \int u(t)f(t)dt + C \]

4. Solve for \[ x(t) \].
Ex. \( x' = x + e^{-t} \).

Step 1: Rewrite it as \( x' - x = e^{-t} \). \( a(t) = 1, f(t) = e^{-t} \).

Step 2: Multiply by the integrating factor \( u(t) = e^{-\int a(t) dt} = e^{-\int 1 \, dt} = e^{-t} \).

so the equation becomes

\[
(e^{-t}x)' = e^{-t}e^{-t}.
\]

Step 3: Integrate this equation to obtain

\[
e^{-t}x(t) = \int e^{-t} \, dt + C' \]

\[
= -\frac{1}{2}e^{-2t} + C
\]

Step 4: \( x(t) = e^{t} \left( -\frac{1}{2}e^{-2t} + C \right) \)

\[
= -\frac{1}{2}e^{-t} + Ce^{t}.
\]
Ex. \( x' = x \cos t + \cos x \). \( x(0) = 2 \).

Step 1: Rewrite this equation as
\[
x' - \cos x = \cos t.
\]
\[
\text{at } t = 0, \ f(t) = \cos t.
\]

Step 2: Multiply by the integrating factor
\[
u(t) = e^{-\int \cos t \, dt} = e^{-\int \cos t \, dt} = e^{-\sin t}
\]
We get,
\[
(e^{-\sin t} x)' = e^{-\sin t} \cos t.
\]

Step 3: Integrate the equation to obtain
\[
e^{-\sin t} x(t) = \int e^{-\sin t} \cos t \, dt + C
\]
\[
= \int e^{-\sin t} d(\sin t) + C
\]
\[
= -e^{-\sin t} + C
\]

Step 4: so \( x(t) = e^{\sin t} (-e^{-\sin t} + C) \)
\[
= Ce^{\sin t} - 1.
\]

For \( C \), plug for \( x(0) = 2 \). \( \Rightarrow 2 = C \cdot e^{\sin 0} - 1 \) \( \Rightarrow 2 = C - 1 \) \( \Rightarrow C = 3 \).

So \( x(t) = 3e^{\sin t} - 1 \). \( \Box \).
Ex of Drawing Direction Field by Human Method:

\[ y' = ty^2. \]

Let \( y' = C. \)

i.e. \( ty^2 = C \)

so \( y^2 = \frac{c}{t} \)

\[ y = \pm \sqrt{\frac{c}{t}}. \]

so \( y = \pm t^{-\frac{1}{2}} \cdot c = 1. \)

Check. \( y(t) = \frac{2}{t^2 + c'} \)

\[ y(t) = \begin{cases} \frac{2}{t^2} & c' = 0, \\ -\frac{2}{t^2 + 2} & c' = -1. \end{cases} \]
Ex. \[ y' = (1+y^2). \]

Ex. \[ y' = 2 - y. \]