Ex. $(1+t^2)y' + 4ty = (1+t^2)^{-2}$, $y(1)=0$.

Step 1: Rewrite it as

$$y' + \frac{4t}{1+t^2}y = (1+t^2)^{-3}$$

then $a(t) = -\frac{4t}{1+t^2}$, $f(t) = (1+t^2)^{-3}$.

Step 2: Multiply by the integration factor $u(t) = e^{-\int a(t)\,dt} = e^{-\int -\frac{4t}{1+t^2} \,dt}$

$$= e^{\int \frac{4t}{1+t^2} \,dt} = e^{2\ln(1+t^2)} = (1+t^2)^2.$$

(PS: If use $s = t^2$, $ds = 2tdt$,

$$\int \frac{4t}{1+t^2} \,dt = \int \frac{2(2tdt)}{1+t^2} = \int \frac{2ds}{1+s} = 2\ln(1+s) = 2\ln(1+t^2).$$

We get

$$\left((1+t^2)^2 y(t)\right)' = (1+t^2)^2 (1+t^2)^{-3} = (1+t^2)^{-1}$$

Step 3: Integrate the equation to obtain

$$(1+t^2)^2 y(t) = \int (1+t^2)^{-1} \,dt = \arctan t + C$$

Step 4: Solve $y(t) = (1+t^2)^{-2} \arctan t + C (1+t^2)^{-2}$.

For $C$, plug in $y(1)=0$, then $0 = (1+1)^{-2} \arctan 1 + C (1+1)^{-2}$

$$= \frac{\pi}{4} + \frac{C}{4}, \quad \Rightarrow C = -\frac{\pi}{2}.$$
Structure Of the Solution

For \( y' = a(t)y + f(t) \). \( \star \)

and \( y_p \) is one solution of it.

Consider the homogeneous equation

\[ y' = a(t)y \] \( \Delta \)

and \( y_h \) is one solution of it.

then \( y_p + Ay_h \) is solution of \( \star \) for every \( A \).

Why? Let's check. plug in \( y_p + Ay_h \).

Left side of \( \star \) = \((y_p + Ay_h)' = y'_p + Ay'_h\)

Since \( y_p \) is solution of \( \star \), \( y'_p = a(t)y_p + f(t) \)

\( y_h \) is solution of \( \Delta \), \( y'_h = a(t)y_h \)

so = \( a(t)y_p + f(t) + Aa(t)y_h \)

= \( a(t)(y_p + Ay_h) + f(t) \).

= Right side of \( \star \). \( \square \)
Summarize this as a fact into a theorem.

Then, suppose that \( y_p \) is a particular solution to the inhomogeneous equation \( y' = a(x)y + f(x) \), and that \( y_h \) is a particular solution to the associated homogeneous equation.

Then every solution to the inhomogeneous equation is of the form

\[ y(t) = y_p(t) + Ay_h(t), \]

where \( A \) is an arbitrary constant.
Indeed, every solution to (*), \( y' = a(t) y + f(t) \), will be \( Y_p + A Y_h \), for some \( A \).

Why? Let’s check.

If \( Y_q \) is another solution to (*),

so \( Y_q' = a(t) Y_q + f(t) \) \( \circ \)

\( Y_q' = a(t) Y_q + f(t) \) \( \circ \)

Use (2) - (1),

get \( (Y_q - Y_p)' = a(t)(Y_q - Y_p) \).

hence \( Y_q - Y_p \) is solution of (a).

so \( Y_q - Y_p = A Y_h \), for some \( A \).

then \( Y_q = Y_p + A Y_h \).
Differential Forms and Differential Equations

We will consider more general equations

\[ P(x,y) + Q(x,y) \, y' = 0 \quad (\Delta) \]

Rewrite it as

\[ P(x,y) + Q(x,y) \, \frac{dy}{dx} = 0. \]

\[ \Rightarrow P(x,y) \, dx + Q(x,y) \, dy = 0. \quad (*) \]

Defn. A differential form in two variables \( x \) and \( y \) is an expression of the type

\[ w = P(x,y) \, dx + Q(x,y) \, dy. \]

Ex. \( 2 \, dx \), \( 2 \, dx + y \, dy \), \( x \, dx \).

So \( y(x) \) is solution to \( P(x,y) + Q(x,y) \, y' = 0 \)

\[ \iff \]

Differential form \( P(x,y) \, dx + Q(x,y) \, dy = 0 \)

For this reason, we will consider \( (*) \) as another way of writing the differential equation in \( (\Delta) \).
Ex. Consider the differential equation
\[ x \, dx + y \, dy = 0 , \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y} . \]
Show that this equation has solutions defined implicitly by the equation \( x^2 + y^2 = C \).

Check: \( y = \pm \sqrt{C - x^2} \).

Just check \( y = \sqrt{C - x^2} \Rightarrow (C - x^2)^{\frac{1}{2}} \)

\[ y'(x) = \frac{1}{2} \cdot (C - x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{\sqrt{C - x^2}} = -\frac{x}{y} . \quad \square \]

Defn. Suppose that solutions to the differential equation \((\Delta)\) are given implicitly by the equation
\[ F(x, y) = C . \]
Then the level sets defined by \( F(x, y) = C \) are called \underline{integral curves} of the differential equation.

Ex. Still same example as above.
\[ x^2 + y^2 = C \] is the integral curve.
Ex. 1. \( M = \sin(x+y) \, dx + (2y + \sin(x+y)) \, dy \), exact.

\[ P = \sin(x+y), \quad Q = 2y + \sin(x+y). \]

\[ \frac{\partial P}{\partial y} = \cos(x+y) \]

\[ \frac{\partial Q}{\partial x} = \cos(x+y) \quad \checkmark \]

2. \( M = \frac{x}{\sqrt{x^2+y^2}} \, dx + \frac{y}{\sqrt{x^2+y^2}} \, dy \), exact.

\[ P = \frac{x}{\sqrt{x^2+y^2}}, \quad Q = \frac{y}{\sqrt{x^2+y^2}}. \]

\[ \frac{\partial P}{\partial y} = -\frac{1}{2} x (x^2+y^2)^{-\frac{3}{2}} \cdot 2y = -xy (x^2+y^2)^{-\frac{3}{2}} \]

\[ \frac{\partial Q}{\partial x} = -\frac{1}{2} y (x^2+y^2)^{-\frac{3}{2}} \cdot 2x = -xy (x^2+y^2)^{-\frac{3}{2}} \quad \checkmark \]

3. \( M = (2x + 3y) \, dx + xy \, dy \), is not exact.

\[ P = 2x + 3y, \quad Q = xy \]

\[ \frac{\partial P}{\partial y} = \frac{3}{y}, \quad \frac{\partial Q}{\partial x} = y \quad \checkmark \]
Solving exact differential equations.

If the equation \( P(x,y) \, dx + Q(x,y) \, dy = 0 \) is exact, the solution is given by \( F(x,y) = C \) where \( F \) is found by solving \( \frac{\partial F}{\partial x} = P \) and \( \frac{\partial F}{\partial y} = Q \) using the steps:

1. Solve \( \frac{\partial F}{\partial x} = P \) by integration:
   \[
   F(x,y) = \int P(x,y) \, dx + \phi(y).
   \]

2. Solve \( \frac{\partial F}{\partial y} = Q \) by choosing \( \phi \) so that
   \[
   \frac{\partial F}{\partial y} = \frac{\partial}{\partial y} \left( \int P(x,y) \, dx + \phi(y) \right) = Q.
   \]

Example: \( \sin(x+y) \, dx + (2y + \sin(x+y)) \, dy = 0 \).

Here \( P(x,y) = \sin(x+y) \), \( Q(x,y) = 2y + \sin(x+y) \).

Step 1: \( \frac{\partial F}{\partial x} = P(x,y) = \sin(x+y) \) by integration:
   \[
   F(x,y) = \int \sin(x+y) \, dx + \phi(y) = -\cos(x+y) + \phi(y).
   \]

Step 2: \( \frac{\partial F}{\partial y} = \sin(x+y) + \phi'(y) = 2y + \sin(x+y) = Q(x,y) \)
   \[
   \Rightarrow \phi'(y) = 2y \quad \Rightarrow \quad \phi(y) = y^2.
   \]
   So \( F(x,y) = y^2 - \cos(x+y) \).
Solutions.

Let's carefully examine what it means when we say the equation \( F(x, y) = C \) gives the general solution to \( u = P(x, y) \, dx + Q(x, y) \, dy = 0 \).

Suppose \( F(x, y) = C \). \( y \) is function of \( x \).

Do differential \( \frac{d}{dx} \) on both sides.

\[
\frac{d}{dx} F(x, y(x)) = 0
\]

So

\[
\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0.
\]

\[\Rightarrow \]

\[
\frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy = 0.
\]

Since \( P \, dx + Q \, dy = 0 \).

So

\[
\frac{\partial F}{\partial x}/P = \frac{\partial F}{\partial y}/Q = \mu = \mu(x, y).
\]

Special Case: \( \mu = 1 \).

\[
w = P(x, y) \, dx + Q(x, y) \, dy = \frac{\partial F}{\partial x} \, dx + \frac{\partial F}{\partial y} \, dy = dF.
\]

called differential of \( F \).
Defn. A differential form \( w = P(x,y) \, dx + Q(x,y) \, dy \) is \( \square \) exact \( \square \)
if it is the differential of a continuously differential function
i.e. \( \exists \, F = F(x,y) \) s.t
\[
P(x,y) = \frac{\partial F}{\partial x}, \quad Q(x,y) = \frac{\partial F}{\partial y}.
\]

Q: 1. Given a differential form \( w = P \, dx + Q \, dy \),
how do we know if it is exact?
2. If \( w \) is exact, is there a way to find \( F \)
such that \( \partial F = P \, dx + Q \, dy \)?

Thm. For \( w = P(x,y) \, dx + Q(x,y) \, dy \),
(a) If \( w \) is exact, then \( \partial P/\partial y = \partial Q/\partial x \).
(b) If \( \partial P/\partial y = \partial Q/\partial x \) holds in a rectangle \( R \),
then \( w \) is exact in \( R \).

Pf: Only for (a). \( w \) is exact \( \iff \) \( P = \partial F/\partial x, \quad Q = \partial F/\partial y \)
\[
\frac{\partial P}{\partial y} = \frac{\partial (\partial F/\partial x)}{\partial y} = \frac{\partial^2 F}{\partial x \partial y}, \quad \frac{\partial Q}{\partial x} = \frac{\partial (\partial F/\partial y)}{\partial x} = \frac{\partial^2 F}{\partial x \partial y}.
\]

So \( \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \).

For (b), see the book.
Here \( \mu' , \mu \) are not depending on \( y \),
we need \( \frac{1}{x} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \) also not depending on \( y \).

So \( \mu = e^{\int \frac{1}{x} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) dx} \) is one solution.

Let’s summarize these results.

The form \( Pdx + Qdy \) has an integrating factor depending on one of the variables under the following conditions:

- If \( h = \frac{1}{x} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \) is a function of \( x \) only,
  then \( \mu(x) = e^{\int h(x) dx} \) is an integrating factor.

(Another case) If \( g = \frac{1}{y} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \) is a function of \( y \) only,
  then \( \mu(y) = e^{-\int g(y) dy} \) is an integrating factor.

Ex. \( u = xy \, dx + x^2 \, dy = 0 \).

\( P = xy \), \( Q = x^2 \),

\( \frac{\partial P}{\partial y} = x \),

\( \frac{\partial Q}{\partial x} = 2x \),

so \( u \) is not exact.
\[ h = \frac{1}{\alpha} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = -\frac{1}{x^2} (x - 2x) = -\frac{1}{x}. \] only depends on \( x \).

So \( \mu(x) = e \int h(x) \, dx = e^{-\ln x} = \frac{1}{|x|} \).

take \( \mu(x) = \frac{1}{x} \).

so
\[
\mu(x) \left( P \, dx + Q \, dy \right) = \frac{1}{x} \left( xy \, dx + x^2 \, dy \right) = y \, dx + x \, dy = d(xy).
\]

so \( f(xy) = xy = C \).

so \( y(x) = \frac{C}{x} \).

Homogeneous Equations

Defn. A function \( G(x, y) \) is homogeneous of degree \( n \) if
\[
G(tx, ty) = t^n G(x, y).
\]
for all \( t > 0 \) and all \( x \neq 0, \ y \neq 0 \).

Ex. \( \frac{1}{x^2+y^2} \) is homogeneous of degree \(-2\).

\( \ln(y/x) \) is homogeneous of degree \( 0 \).

\( x-y-2 \) is not homogeneous.
Ex. \((x^2 + y^2) \, dx - 2xy \, dy = 0\)

Step 1: \( P = x^2 + y^2, \quad Q = -2xy \)

\[
\frac{\partial P}{\partial y} = 2y, \quad \frac{\partial Q}{\partial x} = -2y.
\]

How to find \( \mu \) such that \( \mu \left( (x^2 + y^2) \, dx - 2xy \, dy \right) \) is exact.

So we need to discuss more methods to find \( \mu \).

Integrating factors depending on only one variable.

In fact, we want to find \( \mu \) such that

\[
\mu \cdot m = \mu \cdot P + \mu' \cdot Q \quad \text{is exact.}
\]

i.e.

\[
\frac{2}{\partial y} (\mu P) = \frac{2}{\partial x} (\mu Q).
\]

It is a partial differential equation for \( \mu \).

There is no procedure for solving this equation in general.

However, special case: \( \mu \) only depend on one variable.

Like, \( \mu \) does not depend on \( y \), i.e. \( \mu = \mu(x) \).

So \((*)\) becomes

\[
\mu \frac{\partial P}{\partial y} = \mu \frac{\partial Q}{\partial x} + \mu' \cdot Q.
\]

So

\[
\mu' = \frac{1}{\mu} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) \mu.
\]
Solutions and integrating factors

Now $\mu$ is not necessarily 1.

\[
\frac{\partial F}{\partial x} = \mu P, \quad \frac{\partial F}{\partial y} = \mu Q.
\]

Then $\mu[Pdx + Qdy] = \mu Pdx + \mu Qdy$

\[
= \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy = dF.
\]

So $\mu[Pdx + Qdy]$ is exact.

Here $\mu$ is called an integrating factor.

Now we have a strategy for solving $Pdx + Qdy = 0$.

Step 1: Find an integrating factor $\mu$,

so that $\mu Pdx + \mu Qdy$ is exact.

Step 2: Find a function $F$ such that

\[dF = \mu Pdx + \mu Qdy.\]

then a general solution $y(x)$ to $Pdx + Qdy = 0$

is given implicitly by $F(x, y) = C$.

But the 1st step is very difficult.
Repeat the steps for solving exact forms.

**Step 1:**
\[ F(x, y) = \int u(x, y) \, dx + \phi(y) \]
\[ = \int (1 + \frac{2x}{y}) \, dx + \phi(y) \]
\[ = x + \frac{x^2}{y} + \phi(y) . \]

**Step 2:**
\[ \frac{\partial F}{\partial y} = -\frac{x^2}{y^2} + \phi'(y) = mQ(u, v) = -\frac{x^2}{y^2} . \]

\[ \Rightarrow \phi'(y) = 0 \]
\[ \Rightarrow \phi(y) \equiv C . \]

**Step 3:**
\[ F(x, y) = x + \frac{x^2}{y} + C . \]

so
\[ x + \frac{x^2}{y} + C = C' \]
\[ \Rightarrow x + \frac{x^2}{y} = C \]
\[ \Rightarrow \frac{x^2}{y} = C - x \]
\[ \Rightarrow y = \frac{x^2}{C-x} . \]
Ex. \((y^2+2xy)\,dx - x^2\,dy = 0\).

Step 1: Check whether it is exact!

\[ P = y^2+2xy, \quad Q = -x^2 \]

\[
\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = (2y+2x) - (-2x) = 2y + 4x.
\]

\[
h = \frac{1}{Q} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = -\frac{1}{x^2} (2y+4x) \quad \text{not good.}
\]

\[
g = \frac{1}{P} \left( \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} \right) = \frac{1}{y^2+2xy} (2y+4x) = \frac{2}{y}
\]

is a function only depending on \(y\)! \[\text{good!}\]

Step 2: Using the summary in the box, use \(\mu = e^{-\int g(y)\,dy} = e^{-\int \frac{2}{y}\,dy}
\]

\[
= e^{-2\ln|y|} = \frac{1}{y^2}.
\]

Step 3: Multiply \(\mu\) on both sides.

\[\frac{1}{y^2}(y^2+2xy)\,dx - \frac{x^2}{y^2}\,dy = 0\]

\[\Rightarrow \quad (1+\frac{2x}{y})\,dx - \frac{x^2}{y^2}\,dy = 0\]

\[\text{exact.}\]
Now repeat the steps for solving exact forms.

Step ①: \( F(x, y) = \int \frac{1}{(xy - 1)} \, dx + \phi(y) \)

\[ = \int (y - \frac{1}{x}) \, dx + \phi(y) \]

\[ = xy - \ln|x| + \phi(y) \]

Step ②: \( \frac{\partial F}{\partial y} = x + \phi'(y) = \frac{1}{x^2 - xy} = x - y \)

\[ \implies \phi'(y) = -y \]

\[ \text{so } \phi(y) = -\frac{1}{2}y^2 + C \]

Step ③: so \( F(x, y) = xy - \ln|x| + (-\frac{1}{2}y^2 + C) \)

\[ \text{so } xy - \ln|x| - \frac{1}{2}y^2 + C = C' \]

\[ \text{so } xy - \ln|x| - \frac{1}{2}y^2 = C. \]

\[ \implies y^2 - 2xy + 2\ln|x| + C = 0 \]

\[ \text{so } y = \frac{2x \pm \sqrt{4x^2 - 4(2\ln|x| + C)}}{2} \]

\[ = x \pm \sqrt{x^2 - 2\ln|x| + C}. \quad \Box \]
Ex. \((xy - 1)\, dx + (x^2 - xy)\, dx = 0\)

Step 1: Check whether it's exact.

\[ P = xy - 1, \quad Q = x^2 - xy. \]

\[ \frac{\partial P}{\partial y} = x, \quad \frac{\partial Q}{\partial x} = 2x - y. \]

It's not exact!

\[ \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x} = -x + y. \]

\[ h = \frac{1}{Q} (\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}) = \frac{1}{x^2 - xy} (-x + y) = -\frac{1}{x}. \]

only depending on \(x\).  [Good!!]

Step 2: Using the summary in the box:

use \(\mu = e^{\int h(x)\, dx} = e^{\int -\frac{1}{x}\, dx} = e^{-\ln|x|} = \frac{1}{|x|}.\)

Integrating factor.

use \(\mu = \frac{1}{x}.\)

Step 3: Consider \(\frac{1}{x} (xy - 1)\, dx + \frac{1}{x} (x^2 - xy)\, dy\)

which is exact.
HW 2:
3 2.4, 5, 15, 19
3 2.6. 16, 28

Extra: (1) Draw Direction Field of
\[ x' = \frac{2tx}{1+x}, \]
using Human Method.

(2) And sketch the integral curve of this direction field passing through \( x(0) = 1 \).

(3) Solve the initial value problem
\[ x' = \frac{2tx}{1+x}, \quad x(0) = 1. \]
Draw the curve \( x = x(t) \) on the \( tx \)-plane.
(Rmk: You can compare your answer in (3) with in (2).)