1. Introduction and Background

My research is in the area of ergodic theory and dynamical systems, mainly focused on interval exchange transformations (IET) and symbolical dynamics. In my thesis, I studied two properties of these dynamical systems, one of which is of very recent origin.

Near actions of Polish groups on standard measure spaces and their spatial models have been studied in [MC][RM][VA][B-K]. The existence of Borel lift, i.e. lifting a ‘near action’ group on the σ algebra of measurable sets mod null sets zero to a Borel ‘spatial action’, is an interesting problem. Those actions with pure-point spectrum admit spatial models; on the other hand if an action is weak mixing, then it may not have spatial model. Recently equivalent conditions for a near action to admit a spatial model was raised by E. Glasner, B. Tsirelson, B. Weiss [G-T-W] [G-W]. They also defined the whirly actions and showed that each whirly action admits no nontrivial spatial factor. This is analogous to the relation between weak mixing and pure point spectrum. Since the closure of the set of powers of any automorphism on a standard Lebesgue probability space under the weak topology is a Polish group, the notion of ‘whirly’ for a single automorphism can be defined in terms of the near action of the closure. In [WU], positive results concerning the whirly property are obtained in the context of (1) IET; (2)Del Junco-Rudolph’s map [J-R].

Del Junco-Rudolph’s map is rigid, weak mixing and simple (admitting Property S in the sense of Veech[VE4]). This map is rank one and described by cutting-stacking structure, which is a bridge between the general system and symbolic system. Measure theoretically, almost all the interval exchange transformations are rank one by interval with flat stacks[VE3](for concept of ‘flat stacks’, please see [KI]). In my thesis, the results about the rank one property of the powers of those transformations help us to understand the structures of the corresponding commutants.

Next let me discuss the major results of my thesis:

2. Results on Interval Exchange Transformations

As an active topic in dynamical systems, the theory of interval exchange transformations has been studied by many mathematicians (e.g. [K-S][KE][RZ],[VE1],[VE2],[MS],[BO],[A-F]). Rauzy induction map $T$, acting on the IET space $\Delta_{m-1} \times \mathcal{R}(\pi)$ (where $\Delta_{m-1}$ is the standard $(m-1)$-simplex, $\mathcal{R}(\pi)$ is the Rauzy class of the irreducible $m$-permutation $\pi$), is an important tool for studying the dynamical properties of interval exchange transformations. There exists an absolutely continuous $T$ invariant measure, unique up to a scalar multiple, such that $T$ is ergodic and conservative with respect to this measure[VE1]. By this theory, we can delve into the tower structures of the induced maps associated with Rauzy induction and gain the major results to be described right after.

- A skew product of Rauzy induction with a finite group is constructed and proved to be ergodic and conservative with respect to the product measure. The finite group of the phase space is associated with the return times mod $q$ ($q \in \mathbb{N}$) of the induced map. Thus the following happens for a.e. IET: there exist arbitrary short intervals $I$ such that: (1) the tower with base $I$ has total measure close to the full measure; (2) $(T^q)_I = (T^q)|_I$ (Notation $T|_I$ means the first
return map of \( T \) on \( I \)). Therefore the dominating \( T \) (IET)-tower may be extended to the dominating \( T^q \)-tower. We apply ergodicity and conservativity of Rauzy induction to find:

**Theorem 1** Let \( \pi \) be a nondegenerate \( m \)-permutation. For Lebesgue almost all \( \lambda \in \Delta_{m-1} \) it is true that all powers of the \((\lambda, \pi)\) IET are rank one with ‘flat stacks’.

**Corollary 1** Let \( \pi \) be a nondegenerate \( m \)-permutation. For Lebesgue almost all \( \lambda \in \Delta_{m-1} \) it is true that the commutant of \((\lambda, \pi)\) IET is a perfect Polish group containing a residual set of rank one maps.

- A new criterion for an automorphism to be whirly is given: An ergodic automorphism \( T \) on a standard Lebesgue probability measure space \((\mathcal{X}, \mathcal{B}, \mu, T)\) is whirly iff given \( \varepsilon > 0 \), for any \( l \in \mathbb{N} \) \((-l \in \mathbb{N})\) and any \( \mu \)-positive measure set \( A \in \mathcal{B} \), there exists \( n \in \mathbb{N} \) such that \( T^n \in U_{\varepsilon}(Id) \), and \( \mu(T^n A \cap T^l A) > 0 \). Utilizing Veech’s ergodic theorem about \( T_2 \) on \( \Delta_2 \times \{\pi\} \) \((\pi = (321))[VE1]\), we know essentially all three-IET will visit any open set in \( \Delta_2 \times \{\pi\} \) infinitely often under the iteration of \( T_2 \). Furthermore, the return times \((a_1, a_2, a_3)\) of the induced map associated with the iteration of \( T_2 \) satisfy: \( a_1 + a_3 = a_2 + 1 \). Repeated use of properties of \( T_2 \), together with a density point argument, enable us to verify the above criterion for \( T \) to be whirly. That is:

**Theorem 2** Let \( \pi = (321) \), for Lebesgue almost all \( \lambda \in \Delta_2 \), the three-IET \((\lambda, \pi)\) are whirly, thus admitting no nontrivial spatial factor.

3. Results on del Junco-Rudolph’s Map

The sequence of heights \( \{h_k\} \) of the towers for the del Junco-Rudolph construction [J-R] is automatically a sequence of rigid times for the del Junco-Rudolph’s map \( T \) \((T_{hk} \rightarrow Id)\). We prove that for any positive integer \( q \), there are infinite many heights which are relatively prime to \( q \). By computations associated with the cutting-stacking structure, it is proved that \( T^q \) is rank one with ‘flat stacks’. The conclusion of Corollary 1 is also true for \( T \).

For \( m \) large enough \( T^{h_{m.t}} \) (where \( t \in \mathbb{N}, h_{m,t} = \sum_{j=m}^{m+t} h_j \)) is also close to identity, thus may also be taken as a ‘rigid time’. And the spacer is in the center of each stacks. Based on the above facts, for each cylinder set \( A \) and integer \( l > 0 \), there is a special subset of \( A \) associated with a large block \( B_k \), such that we can see the name of \( A \) at the same position under the following two actions: (1) shifting the subset in any cylinder set to the left by some rigid time; (2) shifting \( A \) to the right by a fixed integer \( l \) \((l > 0)\). The whirly criterion on arbitrary cylinder set is obtained and a lower bound for the corresponding intersection measure is also given.

4. Further Plans

In this section I will describe the areas and problems I will study in the next period, more or less linked with my thesis work.

- Recently Artur Avila and Giovanni Forni proved that for \( m > 2 \), given any nondegenerate irreducible \( m \)-permutation \( \pi \), the measure theoretical generically \( m \)-IET associated with \( \pi \) is weak mixing [A-F]. Whirly implies weak mixing [G-T-W]. So I am interested in the following problem: when \( m > 3 \), and the permutation \( \pi \) is nondegenerate, is the measure theoretically
generic $(\lambda, \pi)$ IET whirly?

- A construction by R. Gunesch and A.B. Katok [G-K] implies there are weak mixing spatial perfect Polish group actions. Since these groups are not weakly closed, it remains open whether there exist weak mixing rigid automorphisms that are not whirly. It is also interesting to find an example of IET which is weak mixing, rigid but not whirly. To complete the study on the whirly property of del Junco-Rudolph’s map. (It is possible that it is whirly, but if not it would provide the example we want.)

- Oleg Ageev proved that in the group of all automorphisms on a Lebesgue probability measure space, the topological generic automorphism is not simple [AG]. That reminds us Veech’s question [VE3]: ‘is it true that measure theoretically generically all the $m$-IET are simple (with self joining supported on a graph) for a fix irreducible permutation?’. That is a question I want to investigate. Recall from [VE4] that a simple transformation is prime if and only if there exits no compact subgroup in its commutant. So I will also study the structure of the commutant of rigid rank one transformations. At the same time one may ask the question ‘is there any transformation satisfying that all the elements in its commutant are rank one except for the identity map?’ Then the results will show lights on the problem of the existence of roots for the IET.

- ‘Pseudo Anosov’ IET [VE1] (i.e. IET of Periodic Type [S-N]) have been studied in [F-M-N] [B-N] [S-U] [WU]. An example of non-weak-mixing substitution is given in [F-M-N]. Is there any whirly Pseudo Anosov IET? What kind of maximal spectral type may a Pseudo Anosov IET admit? Those are among the problems I’d like to solve in the coming years.

**References**

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