ITERATION OF MAPPING CLASSES AND LIMITS OF WEIL-PETERSSON GEODESICS

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Abstract. Let \( S = S_g \) be a closed surface of genus \( g \) with \( g \geq 2 \), \( \text{Mod}(S) \) be the mapping class group of \( S \) and \( \text{Teich}(S) \) be the Teichmüller space of \( S \) endowed with the Weil-Petersson metric. Fix \( X, Y \in \text{Teich}(S) \). In this paper, we show that for any \( \phi \in \text{Mod}(S) \), there exists a positive integer \( k \) only depending on \( \phi \) such that the sequence of the directions of the geodesics connecting \( X \) and \( \phi^k \circ Y \) is convergent in the visual sphere of \( X \) as \( n \) goes to infinity. Moreover, we will give geometric descriptions for these limit geodesics.

1. Introduction

Let \( S = S_g \) be a compact surface of genus \( g \) with \( g \geq 2 \). The Teichmüller space \( T(S) \) of \( S \) carries many canonical metrics such as the Teichmüller metric, the Weil-Petersson metric and so on. These metrics play important roles in Teichmüller theory. In this paper we will focus on the Weil-Petersson metric.

The Teichmüller space, endowed with the Weil-Petersson metric, is a Kähler manifold [Ahl61]. We denote it by \( \text{Teich}(S) \). The space \( \text{Teich}(S) \) has negative sectional curvature [Tro86, Wol86] and non-positive definite Riemannian curvature operator [Wu14]. Although \( \text{Teich}(S) \) is incomplete [Chu76, Wol75], it is geodesically convex [Wol87]. The completion \( \overline{\text{Teich}}(S) \) of \( \text{Teich}(S) \), also called augmented Teichmüller space, is naturally a CAT(0) space [Wol03]. One can refer to the recent nice book [Wol10] of Wolpert for more details on the Weil-Petersson geometry.

Let \( X \in \text{Teich}(S) \) and \( V_X(S) \) be the visual sphere of \( \text{Teich}(S) \) at \( X \) which is the collection of the directions of \( \text{Teich}(S) \) at \( X \). It is clear that \( V_X(S) \) is homeomorphic to a \((6g - 7)\)-dimensional unit sphere. Since \( \text{Teich}(S) \) is incomplete, there exists geodesics starting at \( X \) with directions in \( V_X(S) \) and going to the boundary of \( \overline{\text{Teich}}(S) \) in finite time. Brock-Masur-Minsky in [BMM10] introduced the ending lamination for an infinite geodesic ray. For certain geodesic ray, the ending lamination uniquely determines the Weil-Petersson geodesic. One can see [BMM10, BMM11, BM] for related problems. The purpose of this article is also to study Weil-Petersson geodesics.

The mapping class group \( \text{Mod}(S) \) of \( S \), which is the group of orientation preserving self-homeomorphism of \( S \) up to isotopy, acts on \( \text{Teich}(S) \) by isometries. Let \( \phi \in \text{Mod}(S) \), we define the translation length \( L_{WP}(\phi) \) of \( \phi \) on \( \text{Teich}(S) \) by

\[
L_{WP}(\phi) := \inf_{X \in \text{Teich}(S)} \text{dist}(X, \phi \circ X)
\]

where \( \text{dist} \) is the induced Weil-Petersson distance on \( \text{Teich}(S) \). We call \( \phi \) is semi-simple if \( L_{WP}(\phi) \) is attained in \( \text{Teich}(S) \), otherwise it is called parabolic. In [DW03, Wol03, Yam01] it was shown that a mapping class \( \phi \in \text{Mod}(S) \) is semi-simple if
and only if $\phi$ is either finite ordered or pseudo-Anosov. In particular, every pseudo-Anosov mapping class $\phi \in \text{Mod}(S)$ has a unique axis, which is a $\phi$-invariant geodesic line in $\text{Teich}(S)$.

Let $X, Y \in T(S)$ and $\Gamma(X,Y)$ be the quasi-Fuchsian Bers simultaneous uniformization of $(X,Y) \in T(S) \times T(\overline{S})$; then $\Gamma(X,Y)$ determines a three dimensional hyperbolic manifold, denoted by $Q(X,Y)$. In [Bro01] it was shown that

**Theorem 1.1** (Brock). Let $\phi \in \text{Mod}(S)$ be a mapping class. Then there is an $s \geq 1$ depending only on $\phi$ and bounded in terms of $S$ so that the sequence $\{Q(\phi^s(X), Y)\}_{i \geq 1}$ converges algebraically and geometrically.

As stated above, the Teichmüller space $\text{Teich}(S)$ is geodesically convex [Wol87], i.e., for any two points $X, Y \in \text{Teich}(S)$, there exists a geodesic in $\text{Teich}(S)$ connecting $X$ and $Y$. Moreover, the geodesic is unique because the sectional curvature of $\text{Teich}(S)$ is negative. We denote the geodesic joining $X$ and $Y$ by $g(X,Y)$. Our first result is analogous to Brock’s theorem.

**Theorem 1.2.** Let $\phi \in \text{Mod}(S)$ be a mapping class and $X, Y \in \text{Teich}(S)$. Then there is an $s \geq 1$ only depending on $\phi$ so that the sequence of the directions of the geodesics $\{g(X, \phi^s(Y))\}_{i \geq 1}$ is convergent in the visual sphere $V_X(S)$ of $X$ as $i$ goes to infinity.

The second aim of this article is to study the limit geodesics in Theorem 1.2. Before providing further context we first recall the Thurston-Nielsen classification of mapping classes [FLP12]. A mapping class is called reducible provided that some power fixes a collection of mutually disjoint simple closed curves in $S$. Reducible classes are analyzed in terms of mappings classes of proper subsurfaces. More precisely, for a reducible mapping class $\phi \in \text{Mod}(S)$ there exists a maximal finite collection of mutually disjoint simple closed curves $\{\alpha_i\}$ and mutually disjoint proper subsurfaces $\{PS_k\} \subset S$ and an integer $s \geq 1$ such that $\phi^k$ is the product of Dehn-twists on $\{\alpha_i\}$ and pseudo-Anosov elements on proper subsurfaces $\{PS_k\} \subset S$. A mapping class is precisely one of: finite-ordered, reducible or pseudo-Anosov [FLP12].

Given a collection of mutually disjoint essential simple closed curves, we connect them pairwisely by a segment of length 1. The result object is called a simplex. Let $\sigma$ be a simplex and denote its vertices by $\sigma^0$. Recall that the stratum $T_{\sigma}$ consists of all hyperbolic surfaces with nodes along the curves in $\sigma^0$ [Mas76, Wol03]. A stratum is a convex subset in $\text{Teich}(S)$ [DW03, Wol03, Yam01]. If $\phi$ is a Dehn-twist on an essential simple closed curve $\alpha$, it was shown in [Bro05] that the limit of the sequence $\{g(X, \phi^i \circ Y)\}_{i \geq 1}$ goes to the stratum $T_{\alpha}$. It is interesting to know whether this property can be generalized to a multi Dehn-twist, which corresponds to the case that $\phi$ is reducible with $L_{WP}(\phi) = 0$ in Theorem 1.2. Our second result is to give a positive answer to this question.

Before stating the theorem, let us recall that in [Wol03] Wolpert proved a compactness theorem for any sequence of geodesics in $\overline{\text{Teich}(S)}$ with uniform bounded lengths (see Proposition 23 in [Wol03]). Lately, in [Yam10] Yamada constructed the so-called Teichmüller-Coxeter development $D(\overline{\text{Teich}(S)}, \iota)$ through introducing an infinite Coxeter reflection group and gluing infinite copies of $\overline{\text{Teich}(S)}$ through the strata. The space $D(\overline{\text{Teich}(S)}, \iota)$ is a complete CAT(0) space [Yam10]. The limit geodesic in Wolpert’s compactness theorem can be well described in $D(\overline{\text{Teich}(S)}, \iota)$.
If $\phi$ is a multi Dehn-twist, the following result tells that the limit of the geodesics \( \{g(X, \phi^n(Y))\}_{i \geq 1} \) exists in some sense. More precisely,

**Theorem 1.3.** Let $\sigma$ be a $m$-simplex and $\sigma^0 = \{\alpha_1, \cdots, \alpha_{m+1}\}$ and $\tau_i$ be a Dehn-twist about the curve $\alpha_i$ for $i = 1, 2, \cdots, m + 1$. Let $\phi = \prod_{1 \leq i \leq m+1} \tau_i \in \text{Mod}(S)$ and $X, Y \in \text{Teich}(S)$, and $g_n$ be the geodesic $g(X, \phi^n \circ Y)$. Then, there exist a positive number $L$ and an associated partition $0 = t_0 < t_1 < \cdots < t_k = L$, and simplices $\sigma_0, \cdots, \sigma_k$ and a piecewise-geodesic

\[
g : [0, L] \to \text{Teich}(S)
\]

with the following properties.

1. $\sigma^0_i \subset \sigma^0_j$, $\sigma^0_i \cap \sigma^0_j = \emptyset$ for $i \neq j$, where $1 \leq i, j \leq m + 1$.

2. $\sigma^0 = \bigcup_{i=1}^k \sigma^0_i$.

3. $g(t_i) \in T_{\sigma_i}$, $i = 1, \cdots, k - 1$, $g(0) = X, g(t_k) = Y$.

4. The geodesic segments $\{g_n([0, t_1])\}$ converge in $\text{Teich}(S)$ to the restriction $g([0, t_1])$, and for each $i = 1, \cdots, k - 1$,

\[
\lim_{n \to +\infty} \text{dist}(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ g_n(t), g(t)) = 0, \quad \text{for } t \in [t_i, t_{i+1}]
\]

where $\tau_{i,n} = \prod_{\alpha \in \sigma_i} \tau_{\alpha}^{-n}$, for $i = 1, \cdots, k - 1$.

5. The piecewise-geodesic $g$ is the unique minimal length path in $\text{Teich}(S)$ joining $g(0)$ to $g(L)$ and intersecting the closures of the strata $T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}}$ in order.

6. The first point $g(t_1)$ on $g([0, L])$, meeting with strata, is the point where the geodesic joining $(1, X)$ and $(\prod_{\alpha \in \sigma^0} \omega_{\alpha}, Y)$ in the Teichmüller-Coxeter development $D(\text{Teich}(S), i)$ firstly meets with the wall. Where we identify $(1, \text{Teich}(S))$ with $\text{Teich}(S)$.

As part of the analysis we also have the following limit result.

**Theorem 1.4.** Let $\sigma$ be a $m$-simplex with $\sigma^0 = \{\gamma_1, \cdots, \gamma_{m+1}\}$, and $\tau_i$ be a Dehn-twist about the curve $\gamma_i$ for $i = 1, 2, \cdots, m + 1$. Let $\phi = \prod_{1 \leq i \leq m+1} \tau_i \in \text{Mod}(S)$ and $g_n$ be the geodesic $g(X, \phi^n \circ Y)$. Then for any $X, Y \in \text{Teich}(S)$ there exists a constant $L(X, Y) > 0$, depending on $X$ and $Y$, such that

\[
\lim_{n \to +\infty} \ell(g_n) = L(X, Y) > 0
\]

where $\ell(g_n)$ is the length of the geodesic $g(X, \phi^n \circ Y)$.

We call a geodesic ray $c : [0, +\infty) \to \text{Teich}(S)$ is strongly asymptotic to a subset $A \subset \text{Teich}(S)$ if the distance between $c(t)$ and $A$ satisfies

\[
\lim_{t \to +\infty} \text{dist}(c(t), A) = 0.
\]

If $\phi$ is pseudo-Anosov in Theorem 1.2, first the results in [DW03, Wol03, Yam01] tell that the $\phi$ is semi-simple and the translation length is positive. The following result tells that the limit geodesic ray of the geodesics $\{g(X, \phi^n \circ Y)\}$ stays in the thick part of the Teichmüller space. Moreover, the length of every essential simple closed curve is unbounded along the limit ray. More precisely,
Theorem 1.5. Let $\phi$ be a pseudo-Anosov mapping class and $X, Y \in \text{Teich}(S)$, then the geodesics \( \{ g(X, \phi^n \circ Y) \} \) converge to a geodesic ray \( c : [0, +\infty) \to \text{Teich}(S) \) which is strongly asymptotic to the axis of $\phi$ in $\text{Teich}(S)$. Moreover, for any essential simple closed curve $\alpha \subset S$,

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty.$$ 

Define the translation length $L_T(\phi)$ of $\phi$ under the Teichmüller metric as $L_T(\phi) := \inf_{X \in \mathcal{T}(S)} \text{dist}_T(X, \phi \circ X)$ where $\text{dist}_T$ is the Teichmüller distance on $T(S)$. Along the axis of $\phi$, the following result tells that the rate of growth of $\ell_\alpha$ is uniformly bounded and independent of the choice of $\alpha$. More precisely,

**Proposition 1.6.** Let $\phi$ be a pseudo-Anosov mapping class and $\gamma : (-\infty, \infty) \to \text{Teich}(S)$ be the axis of $\phi$. Then, for any essential simple closed curve $\alpha \subset S$,

$$\lim_{t \to +\infty} \frac{\ln \ell_\alpha(\gamma(t))}{t} = \frac{L_T(\phi)}{L_{WP}(\phi)} \geq \frac{1}{\sqrt{4\pi(g-1)}}.$$ 

A recent result of Bromberg and Brock in [BB14] implies that the rate $\frac{1}{\sqrt{g}}$ is optimal as $g$ goes to infinity. Actually the Penner’s examples in [Pen91] realize this optimal rate as $g$ goes to infinity. One can see [BB14] for more details.

If $\phi$ is reducible with $L_{WP}(\phi) > 0$ in Theorem 1.2, the following theorem tells that the limit geodesic ray of the geodesics $\{ g(X, \phi^n \circ Y) \}$ goes to an explicit stratum. More precisely,

**Theorem 1.7.** Let $\phi \in \text{Mod}(S)$ be reducible with $L_{WP}(\phi) > 0$ and $k$ be a positive integer such that $\phi^k = \prod_{\alpha \in \sigma_0} \tau_{\alpha} \times \prod_{j} \phi_j$ where $\sigma$ is a simplex and $\tau_{\alpha}$ is Dehn-twist about $\alpha$ and $\phi_j = \phi^k|_{PS_j}$ is pseudo-Anosov on $PS_j$ where $PS_j$ is proper subsurface of $S$. Then for any $X, Y \in \text{Teich}(S)$, there exits a geodesic ray $c : [0, +\infty) \to \text{Teich}(S)$ such that

1. The geodesics $\{ g(X, \phi^n \circ Y) \}$ converge to the ray $c : [0, +\infty) \to \text{Teich}(S)$.
2. For any essential simple closed curve $\alpha \in \partial(\bigcup_j PS_j)$, we have
   $$\lim_{t \to +\infty} \ell_\alpha(c(t)) = 0.$$
3. There exists a positive number $\epsilon_0$, only depending on $X$ and $Y$, such that for any essential simple closed curve $\beta \notin \partial(\bigcup_j PS_j)$, we have
   $$\inf_{t \geq 0} \ell_\beta(c(t)) \geq \epsilon_0.$$

The theorem above and a result of Wolpert in [Wol08] (also see Theorem 2.9 in Section 2) tell that the limit geodesic ray is strongly asymptotic to an explicit stratum whose nodes consist of the boundary curves in the proper subsurfaces on which certain power of $\phi$ is pseudo-Anosov.

Throughout this paper we always assume that the geodesics have unit speeds.

**Plan of the paper.** We set up some notations and provide necessary preliminaries for this article in Section 2. Section 3 will establish a compactness theorem for the sequence $\{ g(X, \phi^n \circ Y) \}$ when $\phi$ is a multi Dehn-twist, which will be applied to prove Theorem 1.2. In Section 4 we will prove Theorem 1.2, 1.3 and 1.4. Theorem
1.5 and Proposition 1.6 will be proven in 5. And we will finish the proof of Theorem 1.7 in Section 6.

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2. Notations and Preliminaries

2.1. Surfaces. Let $S = S_g$ be a closed surface of genus $g$ with $g \geq 2$. Let $M_{-1}$ denote the space of Riemannian metrics on $S$ with constant curvature $-1$, and $X = (S_g, \sigma |dz|^2) \in M_{-1}$. The group $\text{Diff}_0$, which is the group of diffeomorphisms isotopic to the identity, acts by pull back on $M_{-1}$. The Teichmüller space $T(S)$ of $S$ is defined as

$$T(S) := M_{-1}/\text{Diff}_0.$$  

The Teichmüller space $T(S)$ has a natural complex structure. Its holomorphic cotangent space $T^*_X T(S)$ at $X$ is identified with the quadratic differentials $Q(X) = \varphi(z)dz^2$ on $X$. The Weil-Petersson metric is the Hermitian metric on $T(S)$ arising from the Petersson scalar product

$$<\varphi, \psi> = \int_X \frac{\varphi \cdot \overline{\psi} \; dz \wedge d\overline{z}}{-2i}$$

via duality. We will concern ourselves primarily with its Riemannian part $<,>_WP$. All through this paper we denote the Teichmüller space endowed with the Weil-Petersson metric by Teich($S$). Let $\alpha$ be an essential simple closed curve on $S$, then for any $X \in \text{Teich}(S)$, there exists a unique closed geodesic $[\alpha]$ in $X$ which represents for $\alpha$ in the fundamental group of $S$. We denote the length of $[\alpha]$ in $X$ by $\ell_\alpha(X)$. The following convexity theorem in [Wol87] is crucial in this paper. One can also see an alternative proof in [Wol12] due to M. Wolf.

**Theorem 2.1** (Wolpert). Given an essential simple closed curve $\alpha \subset S$ and a Weil-Petersson geodesic $c(t)$ in Teich($S$), then the length function $\ell_\alpha(c(t))$ is strictly convex along $c(t)$.

2.2. Augmented Teichmüller space. The non-completeness of the Weil-Petersson metric corresponds to finite-length geodesics in Teich($S$) along which the length function of some simple closed curve pinches to zero. In [Mas76] the augmented Teichmüller space is described concretely by adding strata consisting of stratum $T_\sigma$ defined by the vanishing of lengths

$$\ell_\alpha = 0$$

for each $\alpha \in \sigma^0$ where $\sigma^0$ is a collection of finite mutually essential disjoint simple closed curves.

**Definition 2.2.** A $k$-simplex $\sigma$ is a simplex whose vertices $\sigma^0$ is a set of $k+1$ distinct free homotopy classes of essential mutually disjoint simple closed curves of $S$. We call two simplices $\sigma$ and $\eta$ are disjoint if $\sigma^0$ and $\eta^0$ are mutually disjoint.

The topology of the stratum $T_\sigma$ can be described by the so-called extended Fenchel Nielsen coordinates: Give a pants decomposition $P$ with $\sigma^0 \subset P$, the usual coordinates map Teich($S$) to $\prod_{\alpha \in P} \mathbb{R} \times \mathbb{R}^+$, where the first coordinate of
each pair measures twist and the second measures the length of the corresponding simple closed curve in $P$. We extend the second part to 0 and take the quotient by identifying $(t, 0)$ and $(t', 0)$ in each $\mathbb{R} \times \mathbb{R}_{\geq 0}$ factor. The topology near every point in a stratum $T_\sigma$ is given by this extended coordinate.

The stratum $T_\sigma$ are naturally products of lower dimensional Teichmüller spaces corresponding to the nodal surfaces in $T_\sigma$ [Mas76]. So the stratum $T_\sigma$ may contain totally geodesic Euclidean spaces. For the one dimensional flat case, the following theorem is called the non-refraction property for Weil-Petersson geodesic.

**Theorem 2.3** ([DW03, Wol03, Yam04]). Let $g(X, Y)$ be the geodesic connecting $X$ and $Y$ in $\text{Teich}(S)$, and let $\sigma_1^0$ and $\sigma_2^0$ be the maximal collection of essential simple closed curves so that $X \in T_{\sigma_1}$ and $Y \in T_{\sigma_2}$. If $\eta = \sigma_1 \cap \sigma_2$, then

$$\text{int}(g) \subset T_\eta$$

where $\text{int}(g)$ is the interior of the geodesic $g(X, Y)$.

We remark that for a special case that both $X$ and $Y$ lie in $\text{Teich}(S)$, the theorem above is simple a restatement of Wolpert’s geodesical convexity theorem [Wol87].

**2.3. Mapping class group.** Let $\text{Diff}^+$ be all the self-homeomorphisms of $S$ preserving the orientation. The mapping class group $\text{Mod}(S)$ of $S$ is defined by

$$\text{Mod}(S) := \text{Diff}^+ / \text{Diff}^0.$$ 

The mapping class group $\text{Mod}(S)$ acts on $\overline{\text{Teich}}(S)$ as isometries. Actually, the whole isometry group of $\overline{\text{Teich}}(S)$ is almost $\text{Mod}(S)$ [BM07, MW02]. A mapping class is irreducible provided no power fixes the free homotopy class of an essential simple closed curve. An irreducible mapping class is pseudo-Anosov. Thurston’s classification Theorem tells that a mapping class is precisely one of: periodic, irreducible or reducible [FLP12]. For a reducible mapping class $h$ an invariant is $\sigma_h$ the maximal simplex fixed by some power of $h$. Recall the Weil-Petersson translation length is defined as

$$L_{WP}(\phi) = \inf_{X \in \text{Teich}(S)} \text{dist}(X, \phi \circ X)$$

where dist is the intrinsic distance on $\text{Teich}(S)$ induced by the Weil-Petersson metric. The following classification theorem of $\text{Mod}(S)$ in terms of Weil-Petersson translation lengths is crucial in this article.

**Theorem 2.4** ([Wol03]). Let $\phi \in \text{Mod}(S)$. Then we have

(1). The mapping class $\phi$ is semi-simple if and only if $L_{WP}(\phi)$ attains its minima at a point in $\text{Teich}(S)$. In this case either $\phi$ fixes a point in $\text{Teich}(S)$ or there exists a unique bi-infinite Weil-Petersson geodesic $c(t) \subset \text{Teich}(S)$ such that for all $t \in \mathbb{R}$ $\phi \circ c(t) = c(t + L_{WP}(\phi))$. For the later case $\phi$ is pseudo-Anosov and $L_{WP}(\phi) > 0$.

(2). The mapping class $\phi$ is reducible if and only if $L_{WP}(\phi)$ can not attain its minima in $\text{Teich}(S)$. In this case either $\phi$ fixes a stratum in $\overline{\text{Teich}}(S)$ or there exists a positive integer $k$ depending on $\phi$ such that $\phi^k$ acts on the null stratum $S(\sigma)$ which is a product of low-dimensional Teichmüller spaces $\prod(T') \times \prod(T'')$ by a product of : irreducible elements $\phi'$ on $T'$ with axis $r_{\phi'}$ and the identity on each $T''$. For the later case in particular there exists a bi-infinite Weil-Petersson geodesic $c(t) \subset \overline{\text{Teich}}(S)$ such that for all $t \in \mathbb{R}$ $\phi^k \circ c(t) = c(t + kL_{WP}(\phi))$. 
For the irreducible case one can also refer to [DW03, Yam01].

One immediate consequence of Theorem 2.4 is the following result, which will be applied later.

**Proposition 2.5.** Let $\phi \in \text{Mod}(S)$ be a reducible mapping class with $L_{WP}(\phi) = 0$. Then there exist an integer $k$ and a simplex $\sigma$ such that $
abla^k = \prod_{\alpha_i \in \sigma} \tau_{\alpha_i}$, where $\tau_{\alpha_i}$ is a Dehn-twist about $\alpha_i$.

**Proof.** Since $\phi$ is reducible, there exists an integer $k > 0$, a simplex $\sigma$ and a collection of mutually disjoint proper subsurfaces $\{PS_j\}$ such that

$$\phi^k = \prod_{\alpha_i \in \sigma} \tau_{\alpha_i} \cdot \prod_j \phi_j$$

where $\phi_j$ is pseudo-Anosov on $PS_j$ (see [FLP12]).

Since the stratum $S(\sigma)\phi$ is the product of lower dimensional Teichmüller spaces endowed with the Weil-Petersson metric, it is not hard to see that

$$L_{WP}(\phi^k) = \sum_{\alpha_i \in \sigma} (L_{WP}(\tau_{\alpha_i}))^2 + \sum_j (L_{WP}(\phi_j))^2.$$  

Since $\text{Teich}(S)$ is a complete CAT(0) space, elementary CAT(0) geometry (see [BH99]) tells that

$$L_{WP}(\phi^k) = |k| L_{WP}(\phi).$$

That is, the pseudo-Anosov part of $\phi$ does not exist. Hence,

$$\phi^k = \prod_{\alpha_i \in \sigma} \tau_{\alpha_i}.$$

□

2.4. **Weil-Petersson geodesics and Alexandrov tangent cone.** Let $X \in \text{Teich}(S)$ and $WPG_X(S)$ be the set of Weil-Petersson geodesics starting from $X$. Define the map

$$g_X : \overline{\text{Teich}(S)} \to WPG_X(S)$$

which maps $Y \in \overline{\text{Teich}(S)}$ to the unique Weil-Petersson geodesic joining $X$ and $Y$. Since $\text{Teich}(S)$ is a CAT(0) space, the map $g_X$ is well-defined. And it is not hard to see that $g_X$ gives a homeomorphism from $\text{Teich}(S)$ to $WPG_X(S)$.

Let $c_1(t)$ and $c_2(t)$ in $WPG_X(S)$ be two geodesics. As in [BH99], the Alexandrov angle between $c_1(t)$ and $c_2(t)$ at $X$ is defined with values in $[0, \pi]$ as follows

$$\cos \angle(c_1, c_2) = \lim_{t \to 0^+} \frac{2t^2 - \text{dist}^2(c_1(t), c_2(t))}{2t^2}.$$  

Introduce the equivalence relation on $WPG_X(S)$ as $c_1(t) \sim c_2(t)$ provided that $\angle(c_1, c_2) = 0$.

**Definition 2.6.** The Alexandrov tangent cone $AC_X$ at $X$ is defined to be the quotient space $WPG_X(S)/\sim$. 

If $X \in \overline{\text{Teich}(S)}$, the Alexandrov cone $AC_X$ coincides with the tangent space of $\overline{\text{Teich}(S)}$ at $X$, which is the Euclidean space $\mathbb{R}^{6g-6}$. Let $\sigma$ be a simplex and $X \in T_{\sigma}$. Consider a Fricke-Klein basis $\{\text{grad}(\ell_\beta)\}_{\beta \in \lambda}$ for $T_{\sigma}$ around $X$. For a geodesic $r(t)$ with initial point $X$. We define a map $\Lambda : r(t) \to \mathbb{R}^{6g-6} \times T_XT_\sigma$ by

$$\Lambda(r(t)) = \left(\sqrt{2\pi} \frac{d\ell_\alpha}{dt}(r(0^+)), \sqrt{2\pi} \frac{d\ell_\beta}{dt}(r(0^+))\right).$$

Wolpert in [Wol08] characterized the Alexandrov tangent cone at $X$ as follows.

**Theorem 2.7** (Wolpert). For $X \in \overline{\text{Teich}(S)}$, the map $\Lambda : AC_X \to \mathbb{R}^{6g-6} \times T_XT_\sigma$ is an isometry of cones with restrictions of inner products. A geodesic $c(t)$ with $c(0) = X$ and $\frac{d\ell_\alpha}{dt}(c(0^+)) = 0, \alpha \in \sigma^0$, is contained in the stratum $T_\alpha$.

Theorem 2.3 tells that the completion of a stratum is a closed convex subset in $\overline{\text{Teich}(S)}$. Since $\overline{\text{Teich}(S)}$ is a complete CAT(0) space, elementary CAT(0) geometry tells that the nearest projection onto the completion of a stratum is well defined (see Proposition 2.4 in [BH99]). A consequence of Theorem 2.7 is

**Proposition 2.8.** Let $\alpha \subset S$ be an essential simple closed curve and $T_\alpha$ be the corresponding stratum. Consider the nearest projection map

$$\pi : \text{Teich}(S) \to T_\alpha.$$

Then, for any $X \in \text{Teich}(S)$ we have

$$\pi(X) \in T_\alpha.$$

In particular, for any essential simple closed curve $\beta \subset S$ disjoint with $\alpha$ we have

$$\ell_\beta(\pi(X)) > 0.$$

**Proof.** We argue by contradiction. Assume not. That is, there exists an essential simple closed curve $\beta \subset S$ which is disjoint with $\alpha$ with

$$\pi(X) \in \overline{T_{\alpha \cup \beta}}.$$

Consider the geodesic segment $g(\pi(X), X)$. Since $X \in \overline{\text{Teich}(S)}$, Theorem 2.7 tells that the tangent vector $V$ of $g(\pi(X), X)$ at $\pi(X)$ may be given by

$$V = (v_\alpha, v_\beta, v_3) \in AC_{\pi(X)}$$

with $v_\alpha > 0$ and $v_\beta > 0$ where $v_\alpha \in \mathbb{R}^{6g-6}$, $v_\beta \in \mathbb{R}^{6g-6}$ and $v_3 \in \mathbb{R}^{6g-6}$.

Let $\epsilon > 0$ be small enough and choose a geodesic $c : [0, \epsilon) \to \overline{T_\alpha}$ with $c(0) = \pi(X)$ whose tangent vector at $\pi(X)$ satisfies

$$c'(0) = (0, v_\beta, v_3).$$

Theorem 2.7 tells that the geodesic

$$c((0, \epsilon)) \subset \overline{T_\alpha}.$$
Consider the variation of the geodesics \( \{g(X, c(t))\}_{0 \leq t \leq \epsilon} \). The first variation formula for distance function (see [dC92]) tells that
\[
\frac{d(\text{dist}(X, c(t)))}{dt} \bigg|_{t=0} = -\frac{V \cdot c'(0)}{||V|| \times ||c'(0)||} = -\frac{v_3^2 + ||v_3||^2}{||V|| \times ||c'(0)||} < 0
\]
because \( v_3 > 0 \).

In particular, for small enough \( t_0 \in (0, \epsilon) \) we have
\[
\text{dist}(X, c(t_0)) < \text{dist}(X, \pi(X))
\]
which is a contradiction since \( \pi(X) \) is the nearest projection point of \( X \) in \( \overline{T_{\alpha}} \). \( \square \)

2.5. **Distance to a stratum.** Given a simplex \( \sigma \) and \( X \in \text{Teich}(S) \). The following upper bound for the distance from \( X \) to the stratum \( T_{\sigma} \) will be used (see section 4 in [Wol08])

**Theorem 2.9** (Wolpert). \( \text{dist}(X, T_{\sigma}) \leq \sqrt{2\pi} \cdot \sum_{\alpha \in \sigma^0} \ell_{\alpha}(X) \).

A lower bound for the distance from \( X \) to some stratum may be induced from the following result of Wolpert.

**Lemma 2.10** ([Wol08]). The Weil-Petersson pairing of length functions gradients of disjoint geodesics \( \alpha, \beta \) satisfies
\[
0 < \nabla \ell_{\alpha} \cdot \nabla \ell_{\beta} - \frac{2}{\pi} \ell_{\alpha} \delta_{\alpha,\beta} = O(\ell_{\alpha}^2 \ell_{\beta}^2)
\]
where for \( c_0 > 0 \) the remainder term constant is uniform for \( \ell_{\alpha}, \ell_{\beta} \leq c_0 \).

**Proposition 2.11.** Let \( X \in \text{Teich}(S) \) and \( \alpha \) be an essential simple closed curve on \( S \). Assume that there exists a constant \( \epsilon > 0 \) such that \( \ell_{\alpha}(X) \geq \epsilon \). Then there exists a constant \( C(\epsilon) > 0 \), depending on \( \epsilon \), such that
\[
\text{dist}(X, T_{\alpha}) \geq C(\epsilon) > 0.
\]

Proof. For any \( Z \in T_{\alpha} \) and consider the geodesic \( c : [0, \text{dist}(Z, X)] \rightarrow \overline{\text{Teich}(S)} \) of unit speed with \( c(0) = Z \). Let \( t_0 \in [0, \text{dist}(Z, X)] \) be the first time such that for all \( 0 \leq t \leq t_0 \),
\[
$$
\ell_{\alpha}(c(t)) \leq \min\{\epsilon, c_0\}
$$
\]
where \( c_0 \) is the constant in Lemma 2.10.

In particular, we have
\[
\ell_{\alpha}(t_0) = \min\{\epsilon, c_0\}.
\]

Lemma 2.10 tells that all \( 0 \leq t \leq t_0 \),
\[
||\nabla \ell_{\alpha}(c(t))|| \leq \left( \frac{2}{\pi} c_0 + K(c_0) c_0^2 \right)^{\frac{1}{2}}
\]
where \( K(c_0) \) is a constant depending on \( c_0 \).

Thus,
\[
\min\{\epsilon, c_0\} = \int_0^{t_0} ||\nabla \ell_{\alpha}(c(t))|| \, dt \\ \leq t_0 \left( \frac{2}{\pi} c_0 + K(c_0) c_0^2 \right)^{\frac{1}{2}}.
\]
Rewrite it as
\[
\text{dist}(X, Z) \geq t_0 \geq \min \{ \epsilon, c_0 \} \left( \frac{2}{2\pi} c_0 + K(c_0) c_0^2 \right)^{\frac{1}{2}}.
\]
Since \( Z \in T_\alpha \) is arbitrary, the conclusion follows by choosing
\[
C(\epsilon) = \min \{ \epsilon, c_0 \} \left( \frac{2}{2\pi} c_0 + K(c_0) c_0^2 \right)^{\frac{1}{2}}.
\]
\[\square\]

2.6. Flat triangles in \( \overline{\text{Teich}(S)} \).

In \cite{Wol03} the flat subspaces in \( \overline{\text{Teich}(S)} \) was well studied. As an application Wolpert gave a new proof for a theorem of Brock-Farb in \cite{BF06} which says that \( \text{Teich}(S) \) is in general not Gromov-hyperbolic except in several cases. Recall a geodesic triangle \( \Delta \) is flat provided that \( \Delta \) is isometric to a triangle in the two-dimensional Euclidean space \( \mathbb{R}^2 \). The following result follows from Theorem 2.3.

**Proposition 2.12** (\cite{Wol03}, Proposition 16). Let \( \Delta \) be a flat geodesic triangle in \( \overline{\text{Teich}(S)} \). Then there exists a simplex \( \sigma \) such that the interior of \( \Delta \) is contained in the stratum \( T_\sigma \). Moreover, the projection of \( \Delta \) to each component Teichmüller space of \( T_\sigma \) is either a point or a geodesic segment.

Recall that a reducible mapping class \( \phi \in \text{Mod}(S) \) determines a maximal collection of essential simple closed curves \( \{ \alpha_i \} \) and a maximal collection of proper subsurfaces \( \{ PS_j \} \) of \( S \) such that there exists a positive integer \( k \) such that
\[
\phi^k = \left( \prod_i \tau_{\alpha_i} \right) \cdot \left( \prod_j \phi_j \right)
\]
where \( \tau_i \) is a Dehn-twist about \( \alpha_i \) and \( \phi_j = \phi^k |_{PS_j} \) is pseudo-Anosov on \( PS_j \).

The following technical result is modified from the lemma above which will be applied to prove Theorem 1.7.

**Lemma 2.13.** Let \( \phi \in \text{Mod}(S) \) be reducible and \( k \) be an integer such that \( \phi^k = \prod_j \phi_j \) where \( \phi_j = \phi^k |_{PS_j} \) is pseudo-Anosov on \( PS_j \), and let \( \sigma \) be a simplex with \( \sigma^0 = \bigcup_{\beta \in \partial(PS_j) \beta} \) and \( r : (-\infty, +\infty) \to T_\sigma \) be an axis for \( \phi^k \) such that for all \( t \in \mathbb{R} \), \( \phi^k \circ r(t) = r(t + kL_{WP}(\phi)) \).

Then there does not exist any flat geodesic triangle \( \Delta \) in \( \overline{\text{Teich}(S)} \) whose three vertices \( X, Y, Z \) satisfying \( X = r(0), Y = r(kL_{WP}(\phi)) \) and \( Z \in T_\rho \) where \( \rho \) is a simplex satisfying \( \rho^0 \subset \bigcup_j PS_j \) and \( \rho \neq \sigma \).

**Proof.** We argue by contradiction. Assume not. That is, there exists a flat geodesic triangle \( \Delta \subset \overline{\text{Teich}(S)} \) whose three vertices \( X, Y, Z \) satisfying \( X = r(0), Y = r(kL_{WP}(\phi)) \) and \( Z \in T_\rho \) where \( \rho \) is a simplex satisfying \( \rho^0 \subset \bigcup_j PS_j \) and \( \rho \neq \sigma \).

Proposition 2.12 tells that
\[
\rho^0 \cap \sigma^0 \neq \emptyset
\]
otherwise, from Proposition 2.12 we know that the interior of \( \Delta \) is flat in \( \text{Teich}(S) \) which is impossible because the sectional curvature of \( \text{Teich}(S) \) is negative \cite{Tro86, Wol86}. We prove it through the following two cases.

Case (a). \( \rho \) is a proper subsimplex of \( \sigma \).
Since \( \rho^0 \neq \sigma^0 \), there exists a simple closed curve \( \gamma \in (\sigma^0 - \rho^0) \) such that
\[
\ell_{\gamma}(Z) \neq 0.
\]

Let \( S' \) be a component of \( S - \bigcup_{\alpha \in \rho^0} \alpha \) with \( \gamma \subset S' \) and \( \text{Teich}(S') \) be the Teichmüller space of \( S' \) endowed with the Weil-Petersson metric. Then \( \overline{\text{Teich}(S')} \) is a closed totally geodesic subspace in \( \overline{\text{Teich}(S)} \). Consider the nearest projection
\[
\pi : \Delta \rightarrow \overline{\text{Teich}(S')}.
\]

It is clear that \( \pi(g(X,Y)) \subset \partial \overline{\text{Teich}(S')} \) since \( \rho^0 \subset \sigma^0 \). From equation (3) and Proposition 2.8 we know that that
\[
\pi(Z) \in \text{Teich}(S').
\]

Elementary CAT(0) geometry (see [BH99]; the nearest projection is distance non-increasing) tells that the projection \( \pi(\Delta) \) is a flat geodesic triangle in \( \overline{\text{Teich}(S')} \) whose vertices are \( \{ \pi(X), \pi(Y), \pi(Z) \} \) where \( g(\pi(X), \pi(Y)) \subset \partial \overline{\text{Teich}(S')} \) and \( \pi(Z) \in \text{Teich}(S') \). This is impossible since the sectional curvature of of \( \text{Teich}(S') \) is negative [Tro86, Wol86].

Case (2). There exists a simple closed curve \( \gamma \in \rho^0 \) but \( \gamma \notin \bigcup \partial(PS_j) \).
Since \( \rho^0 \subset \bigcup \partial(PS_j) \), without loss of generality, we may assume that \( \gamma \subset PS_1 \). Let \( \text{Teich}(PS_1) \) be the Teichmüller space of \( PS_1 \) endowed with the Weil-Petersson metric. Then \( \overline{\text{Teich}(PS_1)} \) is a closed totally geodesic subspace in \( \text{Teich}(S) \). Consider the nearest projection
\[
\pi : \Delta \rightarrow \overline{\text{Teich}(PS_1)}.
\]

Since \( \phi^k|_{PS_1} \) is pseudo-Anosov and the geodesic \( g(X,Y) \) is on the axis of \( \phi^k \), it is clear that the projection \( \pi(g(X,Y)) \) is on the axis of \( \phi^k|_{PS_1} \) in \( \text{Teich}(PS_1) \).

On the other hand, since \( \gamma \subset PS_1 \), the projection point \( \pi(Z) \) satisfies
\[
\ell_{\gamma}(\pi(Z)) = 0.
\]

Thus, the projection \( \pi(\Delta) \) is a flat geodesic triangle in \( \overline{\text{Teich}(PS_1)} \) whose vertices are \( \{ \pi(X), \pi(Y), \pi(Z) \} \) where \( \pi(Z) \in \partial \overline{\text{Teich}(PS_1)} \) and \( g(\pi(X), \pi(Y)) \subset \text{Teich}(PS_1) \). This is impossible since the sectional curvature of of \( \text{Teich}(PS_1) \) is negative [Tro86, Wol86].

\[\square\]

### 2.7. Piecewise-geodesics in \( \text{Teich}(S) \)

The following strong reflection property was proved in the paragraph after Theorem 6.9 in Chapter 6 of [Wol10] (or the paragraph after Example 4.19 in [Wol08]). We enclose this section by this property.

**Proposition 2.14** (Wolpert). *Given a simplex \( \sigma \) and let \( X, Y \) be two points in \( \text{Teich}(S) \). Let \( Z \) be a point in \( T_{\sigma} \) such that the piecewise-geodesics \( g(X,Z) \cup g(Z,Y) \) is the minimal length path in \( \text{Teich}(S) \) joining \( X \) and \( Y \) and intersecting the stratum \( T_{\sigma} \). Then, we have

1. The initial tangents of \( g(Z,X) \) and \( g(Z,Y) \) at \( Z \) are equal in the components \( R^{[\sigma]} \).
2. The sum of the initial tangents of \( g(Z,X) \) and \( g(Z,Y) \) at \( Z \) has vanishing projection into the subcone \( T_Z T_{\sigma} \).*
3. Iteration on multi Dehn-twists

Let \( \alpha \) be an essential simple closed curve on \( S \) and \( T_\alpha \) be the \( \alpha \)-stratum in \( \text{Teich}(S) \). Let \( \tau_\alpha \in \text{Mod}(S) \) be a Dehn-twist about \( \alpha \). We start with the following lemma which was proved in [Bro05] by using the well-known Collar Lemma.

**Lemma 3.1** (Brock). Let \( X \) and \( Y \) lie in \( \text{Teich}(S) \) and \( \alpha \) be an essential simple closed curve on \( S \). Then the initial segments of the geodesics \( \{ g(X, \tau_\alpha^n \circ Y) \} \) converge in \( \overline{\text{Teich}(S)} \) to a geodesic \( g(X, Z) \) with a point \( Z \in T_\alpha \). The complementary geodesic segments of \( \tau_\alpha^{-n} \circ g(X, \tau_\alpha^n \circ Y) \) converge in \( \overline{\text{Teich}(S)} \) to a geodesic \( g(Z, Y) \). The concatenation \( g(X, Z) \cup g(Z, Y) \) is the unique shortest path in \( \text{Teich}(S) \) with given endpoints and intersecting \( T_\alpha \).

One can also see the proof of Lemma 3.1 in [Wol10]. The aim of this section is to generalize Lemma 3.1 to certain extents when \( \alpha \) is a multi-curve. These generalizations will be applied to prove Theorem 1.2 and 1.3 in Section 4.

In a complete Riemannian manifold the Arzelà-Ascoli theorem implies that every sequence of geodesics with the same initial point of uniformly bounded lengths has convergent subsequences. The local compactness property for a complete Riemannian manifold is required when the Arzelà-Ascoli theorem is applied. Although the completion \( \overline{\text{Teich}(S)} \) is not locally compact, the following compactness theorem of Wolpert gives a version of Arzelà-Ascoli theorem in the Weil-Petersson setting. This compactness theorem will be used several times in this article.

**Theorem 3.2** ([Wol03], Proposition 23). Consider a sequence of geodesics \( \{ g_n \} \) with the same initial point, lengths converging to \( L > 0 \) and parameter intervals converging to \([0, L]\). Then there exist an associated partition \( 0 = t_0 < t_1 < \cdots < t_k = L \) of the interval, and simplices \( \sigma_0, \cdots, \sigma_k \) and simplices \( \nu_i = \sigma_i \cap \sigma_{i-1} \) and a piecewise-geodesic

\[
g : [0, L] \to \overline{\text{Teich}(S)}
\]

with the following properties.

1. \( g(t_{i-1}, t_i) \subset T_{\nu_i}, \ i = 1, \cdots, k. \)
2. \( g(t_i) \in T_{\sigma_i}, \ i = 1, \cdots, k. \)
3. There are elements \( \tau_{i,n} \in Tw(\sigma_i - \nu_i \cup \nu_{i+1}), \) for \( i = 1, \cdots, k - 1 \) so that after passing to a subsequence, \( g_n([0, t_i]) \) converges in \( \overline{\text{Teich}(S)} \) to the restriction \( g([0, t_i]) \) and for each \( i = 1, \cdots, k - 1 \) and \( t \in [t_i, t_{i+1}] \),

\[
\lim_{n \to +\infty} \text{dist}(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ (g_n(t)), g(t)) = 0.
\]

4. The elements \( \tau_{i,n} \) are either trivial or unbounded.

5. The piecewise-geodesic \( g \) is the unique minimal length path in \( \overline{\text{Teich}(S)} \) joining \( g(0) \) to \( g(L) \) and intersecting the closures of the strata \( T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_k} \) in order.

Let \( \phi \in \text{Mod}(S) \) be a multi Dehn-twist about \( \sigma^0 \) where \( \sigma \) is a simplex. Let \( X, Y \in \text{Teich}(S) \) and \( W \in T_\sigma \). So we have \( \phi \circ W = W \). The triangle inequality tells
that for all \( n \geq 1, \)
\[
\text{dist}(X, \phi^n \circ Y) \leq \text{dist}(X, W) + \text{dist}(W, \phi^n \circ Y)
\]
\[
= \text{dist}(X, W) + \text{dist}(W, Y)
\]
\[
< +\infty.
\]

That is, the geodesics \( \{g(X, \phi^n \circ Y)\} \) have uniformly bounded lengths. So Theorem 3.2 can be applied to the sequence \( \{g(X, \phi^n \circ Y)\} \).

**Proposition 3.3.** Let \( \sigma \) be a \( k \)-simplex and \( \sigma^0 = \{\alpha_1, \cdots, \alpha_{k+1}\} \) and \( \tau_i \) be a Dehn-twist about the curve \( \alpha_i \) for \( i = 1, 2, \cdots, k + 1 \). Let \( \phi = \prod_{1 \leq i \leq k+1} \tau_i \in \text{Mod}(S) \). Then, for any \( X, Y \in \text{Teich}(S) \),

(1). There exists a constant \( \epsilon = \epsilon(X,Y) > 0 \), only depending on \( X \) and \( Y \), such that for any essential simple closed curve \( \beta \notin \sigma^0 \) on \( S \) we have
\[
\inf_{Z \in g(X, \phi^n \circ Y)} \ell_{\beta}(Z) > \epsilon.
\]
for all \( n \geq 1 \).

(2). Any limit of geodesics \( \{g(X, \phi^n \circ Y)\} \) in the sense of Theorem 3.2 goes to a stratum \( T_{\sigma_1} \), where \( \sigma_1 \subset \sigma \).

(3). There exists a point \( X^i_n \) on the geodesic \( g(X, \phi^n \circ Y) \) such that
\[
\lim_{n \to +\infty} \text{dist}(X^i_n, T_{\alpha_i}) = 0
\]
for all \( 1 \leq i \leq k + 1 \).

**Proof.** Proof of Part (1). Let \( \beta \) be an essential simple closed curve on \( S \) satisfying \( \beta \notin \sigma^0 \). We separate the proof of Part (1) into the following two cases.

Case (a). If \( \beta \) is disjoint with \( \sigma^0 \). So \( \phi \) fixes \( \beta \). Thus, we have for all \( n \geq 1, \)
\[
\ell_{\beta}(\phi^n \circ Y) = \ell_{\beta}(Y).
\]
We can always find another essential simple closed curve \( \beta' \subset S \) such that
\[
\beta' \notin \sigma^0 \text{ and } \beta' \cap \beta \neq \emptyset.
\]
From Theorem 2.1 we have for all \( n \geq 1 \) and \( Z \in g(X, \phi^n \circ Y), \)
\[
\ell_{\beta'}(Z) \leq \max\{\ell_{\beta'}(X), \ell_{\beta'}(Y)\} < \infty.
\]
Since \( \beta' \cap \beta \neq \emptyset \), the Collar Lemma (see [Bus10]) tells that there exists a positive number \( \epsilon_1 = \epsilon_1(X,Y) \), only depending on \( X \) and \( Y \), such that all \( n \geq 1 \) and \( Z \in g(X, \phi^n \circ Y), \)
\[
\ell_{\beta}(Z) \geq \epsilon_1 > 0.
\]
Case (b). If \( \beta \) intersects at least one of \( \sigma^0 \). Let \( \alpha \in \sigma^0 \) such that
\[
\beta \cap \alpha \neq \emptyset.
\]
First since \( \phi \) fixes \( \sigma^0 \) pointwisely, in particular, for all \( n \geq 1 \) we have
\[
\ell_{\alpha}(\phi^n \circ Y) = \ell_{\alpha}(Y).
\]
From Theorem 2.1 we have for all \( n \geq 1 \) and \( Z \in g(X, \phi^n \circ Y), \)
\[
\ell_{\alpha}(Z) \leq \max\{\ell_{\alpha}(X), \ell_{\alpha}(Y)\} < \infty.
\]
Since \( \beta \cap \alpha \neq \emptyset \), the Collar Lemma (see [Bus10]) tells that there exists a positive number \( \epsilon_2 = \epsilon_2(X, Y) \), only depending on \( X \) and \( Y \), such that for all \( n \geq 1 \) and \( Z \in g(X, \phi^n \circ Y) \),

\[ \ell_{\beta}(Z) \geq \epsilon_2 > 0. \]

Then Part (1) follows by choosing \( \epsilon = \min\{\epsilon_1, \epsilon_2\} \).

**Proof of Part (2).** First any limit of the geodesics \( \{g(X, \phi^n \circ Y)\} \) in the sense Theorem 3.2 goes to a stratum as \( n \) goes to infinity. Otherwise, from Theorem 3.2, after passing to a subsequence, the geodesics \( \{g(X, \phi^n \circ Y)\} \) converge to a geodesic in \( \text{Teich}(S) \). In particular, the points \( \{\phi^n \circ Y\}_{n \geq 1} \), after passing to a subsequence, converge to a point \( W \in \text{Teich}(S) \). Let \( \epsilon_0 > 0 \) be small enough such that the geodesic ball

\[ B(W; \epsilon_0) := \{V; \text{dist}(V, W) \leq \epsilon_0\} \subset \text{Teich}(S). \]

Since \( \lim_{n \to \infty} \phi^n \circ Y = W \), we have for large enough \( n \geq 1 \),

\[ \phi^n \circ B(W; \epsilon_0) \cap B(W; \epsilon_0) \neq \emptyset. \]

This is impossible because the mapping class group acts properly on \( \text{Teich}(S) \) and \( \phi \) is reducible of infinite order. Then Part (2) follows from Part (1) and Theorem 3.2.

**Proof of Part (3).** We argue by induction on \( k \).

If \( k = 0 \), it directly follows from Lemma 3.1.

Assume that the conclusion holds for any \( k < k_0 \).

When \( k = k_0 \). Assume not. Without loss of generality, after passing to a subsequence, we assume that there exists a positive number \( \epsilon_0 \) such that the distance between two sets satisfies

\[ \text{dist}(g(X, \phi^n \circ Y), T_{\alpha_1}) \geq \epsilon_0 > 0 \]

for all \( n \geq 1 \).

We apply Theorem 3.2 to the geodesics \( \{g(X, \phi^n \circ Y)\} \). Part (1) and Part (2) tell that the geodesics \( \{g(X, \phi^n \circ Y)\} \), after passing to a subsequence, converge to a stratum \( T_{\eta_1} \) with

\[ \eta_1^0 \subset \sigma^0. \]

From our assumption on inequality (4), without loss of generality, we may assume that \( \eta_1^0 = \{\alpha_2, \ldots, \alpha_{k_1}\} \) and \( \eta_2^0 = \{\alpha_1, \alpha_{k_1+1}, \ldots, \alpha_{k+1}\} \). Part (3) of Theorem 3.2 gives that there exists a point \( p_1 \in T_{\eta_1} \) such that the piecewise-geodesics \( \{g(X, p_1) \cup g(p_1, \phi^n \circ Y)\} \) and the geodesics \( \{g(X, \phi^n \circ Y)\} \) will coincide as \( n \to \infty \). That is,

\[ \lim_{n \to \infty} \sup_{p \in g(X, p_1) \cup g(p_1, \phi^n \circ Y)} \text{dist}(p, g(X, \phi^n \circ Y)) = 0. \]

Since \( \phi \in \text{Mod}(S) \) and \( \text{Mod}(S) \) acts on \( \text{Teich}(S) \) by isometries,

\[ g(p_1, \phi^n \circ Y) = (\omega_1^n \circ p_1, \phi^n \circ Y) = \omega_1^n \circ g(p_1, \omega_2^n \circ Y) \]

where \( \omega_1 = \prod_{\gamma \in \eta_1^0} \tau_\gamma \) and \( \omega_2 = \prod_{\gamma \in \eta_2^0} \tau_\gamma \).

For any \( \epsilon > 0 \) we let \( p_1^\epsilon \in \text{Teich}(S) \) with \( \text{dist}(p_1^\epsilon, p_1) = \epsilon \). Since \( \alpha_1 \in \eta_1^0 \) and the cardinality \( |\eta_2^0| < k_0 \), from our assumption there exists a point \( X_n^\epsilon \in g(p_1^\epsilon, \omega_2^n \circ Y) \) such that

\[ \lim_{n \to +\infty} \text{dist}(X_n^\epsilon, T_{\alpha_1}) = 0. \]
Since Teich(S) is a CAT(0) space, from Proposition 2.2 of Chapter II.2 in [BH99] we know that there exists \(X_{n,\varepsilon} \in g(p_1, \omega_2^n \circ Y)\) such that
\[
\text{dist}(X_{n,\varepsilon}, X_n^\varepsilon) \leq \max\{\text{dist}(\omega_2^n \circ Y, \omega_2^n \circ Y), \text{dist}(p_1, p_1^1)\} = \varepsilon.
\]

Hence,
\[
(7) \quad \text{dist}(X_{n,\varepsilon}, T_{\alpha_1}) \leq \text{dist}(X_n^\varepsilon, T_{\alpha_1}) + \text{dist}(X_n^\varepsilon, X_{n,\varepsilon}) + \varepsilon.
\]

After taking a super limit on inequality (7), equation (6) tells that
\[
\lim_{n \to +\infty} \text{dist}(X_{n,\varepsilon}, T_{\alpha_1}) \leq \varepsilon.
\]

Since \(X_{n,\varepsilon} \in g(p_1, \omega_2^n \circ Y)\), \(\omega_1^n \circ X_{n,\varepsilon} \in g(p_1, \phi^n \circ Y)\). Then, equation (5) tells that for \(n \in \mathbb{Z}_{>0}\) large enough there exists \(Y_n \in g(X, \phi^n \circ Y)\) such that
\[
\text{dist}(Y_n, \omega_1^n \circ X_{n,\varepsilon}) < \varepsilon_0/2.
\]

The triangle inequality leads to
\[
(9) \quad \text{dist}(Y_n, T_{\alpha_1}) \leq \text{dist}(Y_n, \omega_1^n \circ X_{n,\varepsilon}) + \text{dist}(\omega_1^n \circ X_{n,\varepsilon}, T_{\alpha_1}) < \varepsilon_0/2 + \text{dist}(X_{n,\varepsilon}, T_{\alpha_1}).
\]

After taking a super limit on inequality (9), inequality (8) tells that
\[
\lim_{n \to +\infty} \text{dist}(Y_n, T_{\alpha_1}) \leq \varepsilon_0/2 + \varepsilon,
\]
which contradicts our assumption on inequality (4) by choosing \(\varepsilon = \varepsilon_0^4/4\) since \(\varepsilon\) is arbitrary.

Our next result is to extend Proposition 3.3 to the case that \(X\) is in a stratum. And the analysis on the Alexandrov cone of Wolpert in Section 2 will be applied to prove the following result.

**Proposition 3.4.** Let \(\sigma\) be a \(k\)-simplex, \(\sigma^0 = \{\alpha_1, \ldots, \alpha_{k+1}\}\) and \(\tau_i\) is a Dehn-twist about the curve \(\alpha_i\) for \(i = 1, 2, \ldots, k+1\). Let \(\phi = \prod_{1 \leq i \leq k+1} \tau_i \in \text{Mod}(S)\) be a multi Dehn-twist. Given a simplex \(\sigma'\) disjoint with \(\sigma\) and two points \(X \in T_{\sigma'}\) and \(Y \in \text{Teich}(S)\), then we have

1. There exists a constant \(\varepsilon = \varepsilon(X, Y) > 0\), only depending on \(X\) and \(Y\), such that for any essential simple closed curve \(\beta \subset S\) with \(\beta \notin \sigma^0 \cup \sigma^{00}\) we have
\[
\inf_{Z \in g(X, \phi^n \circ Y)} \ell_\beta(Z) > \varepsilon
\]
for all \(n \geq 1\).

2. Any limit of geodesics \(\{g(X, \phi^n \circ Y)\}\) in the sense of Theorem 3.2 goes to a stratum \(T_{\sigma_1}\) where \(\sigma_1^0 \subset \sigma^0\).

3. There exists a point \(X_{n,\varepsilon}^i\) on the geodesic \(g(X, \phi^n \circ Y)\) such that
\[
\lim_{n \to +\infty} \text{dist}(X_{n,\varepsilon}^i, T_{\alpha_i}) = 0
\]
for all \(1 \leq i \leq k+1\).
Proof of Part (1). The argument is similar as the proof of Part (1) in Proposition 3.3. Let \( \beta \) be an essential simple closed curve on \( S \) satisfying
\[
\beta \notin \sigma^0 \cup \sigma^0.
\]
We separate the proof of Part (1) into the following two cases.

Case (a). If \( \beta \) is disjoint with \( \sigma^0 \cup \sigma^0 \). So \( \phi \) fixes \( \beta \). Then, we have for all \( n \geq 1 \),
\[
\ell_\beta(\phi^n \circ Y) = \ell_\beta(Y).
\]
We can always find another essential simple closed curve \( \beta' \subset S \) such that
\[
\beta' \notin \sigma^0 \cup \sigma^0 \quad \text{and} \quad \beta' \cap \beta \neq \emptyset.
\]
From Theorem 2.1 we have for all \( n \geq 1 \) and \( Z \in g(X, \phi^n \circ Y) \),
\[
\ell_{\beta'}(Z) \leq \max\{\ell_{\beta'}(X), \ell_{\beta'}(Y)\} < \infty.
\]
Since \( \beta' \cap \beta \neq \emptyset \), the Collar Lemma (see [Bus10]) tells that there exists a positive number \( \epsilon_1 = \epsilon_1(X, Y) \), only depending on \( X \) and \( Y \), such that for all \( n \geq 1 \) and \( Z \in g(X, \phi^n \circ Y) \) we have
\[
\ell_{\beta}(Z) \geq \epsilon_1 > 0.
\]
Case (b). If \( \beta \) intersects at least one of \( \sigma^0 \cup \sigma^0 \). Let \( \alpha \in \sigma^0 \cup \sigma^0 \) with
\[
\beta \cap \alpha \neq \emptyset.
\]
First since \( \phi \) fixes \( \sigma^0 \cup \sigma^0 \) pointwisely, in particular, we have for all \( n \geq 1 \),
\[
\ell_{\alpha}(\phi^n \circ Y) = \ell_{\alpha}(Y).
\]
From Theorem 2.1 we have for all \( n \geq 1 \) and \( Z \in g(X, \phi^n \circ Y) \),
\[
\ell_{\alpha}(Z) \leq \max\{\ell_{\alpha}(X), \ell_{\alpha}(Y)\} < \infty.
\]
Since \( \beta \cap \alpha \neq \emptyset \), the Collar Lemma (see [Bus10]) tells that there exists a positive number \( \epsilon_2 = \epsilon_2(X, Y) \), only depending on \( X \) and \( Y \), such that for all \( n \geq 1 \) and \( Z \in g(X, \phi^n \circ Y) \),
\[
\ell_{\beta}(Z) \geq \epsilon_2 > 0.
\]
Then Part (1) follows by choosing \( \epsilon = \min\{\epsilon_1, \epsilon_2\} \).

Proof of Part (2). First any limit of the geodesics \( \{g(X, \phi^n Y)\} \) goes to a stratum. Otherwise, from Theorem 3.2, pass to a subsequence, the geodesics \( \{g(X, \phi^n \circ Y)\} \) converge to a geodesic in \( \text{Teich}(S) \). In particular, the sequence of points \( \{\phi^n \circ Y\} \), after passing to a subsequence, converges to a point in \( \text{Teich}(S) \), which is impossible because the mapping class group acts properly on \( \text{Teich}(S) \) and \( \phi \) is reducible of infinite order (use the same argument in the proof of Part 2 of Proposition 3.3).

Then we argue by contradiction. Assume not. Part (1) tells that any limit of geodesics \( \{g(X, \phi^n \circ Y)\} \) in the sense of Theorem 3.2 goes to a stratum \( T_\eta \) where \( \eta \) satisfies
\[
\eta \cap \sigma_0 \cup \sigma_0 \quad \text{and} \quad \eta \cap \sigma_0 \neq \emptyset.
\]
We denote the point where the limit geodesic firstly meets with the strata by \( Z \). More precisely, from Theorem 3.2 there exists an associated partition \( 0 = t_0 < t_1 < \cdots < t_k = L \) of the interval, and simplices \( \sigma_0, \cdots, \sigma_k \) and simplices \( \nu_1 = \sigma_1 \cap \sigma_{i-1} \) and a piecewise-geodesic
\[
g : [0, L] \to \text{Teich}(S)
\]
with the following properties.

(a). \( g(0) = X, g(t_1) = Z, g(t_{i-1}, t_i) \subset T_{\nu_i}, g(t_i) \in T_{\sigma_i}, i = 1, \ldots, k. \)

(b). \( \sigma_0 = \sigma', \sigma_1 = \eta, \eta^0 \cap \sigma^0 \neq \emptyset. \)

(c). There are elements \( \tau_{i,n} \in Tw(\sigma_i - \nu_i \cup \nu_{i+1}), \) for \( i = 1, \ldots, k - 1 \) so that after passing to a subsequence, \( g_n([0, t_1]) \) converges in \( \overline{\text{Teich}(S)} \) to the restriction \( g([0, t_1]) \) and for each \( i = 1, \ldots, k - 1 \) and \( t \in [t_i, t_{i+1}], \)

\[
\lim_{n \to +\infty} \dist(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ (g_n(t)), g(t)) = 0.
\]

(d). The piecewise-geodesic \( g \) is the unique minimal length path in \( \overline{\text{Teich}(S)} \) joining \( g(0) \) to \( g(L) \) and intersecting the closures of the strata \( T_{\sigma_1}, T_{\sigma_2}, \ldots, T_{\sigma_k} \) in order.

From equation (10) there exists an essential simple closed curve \( \beta \subset S \) with \( \beta \in \eta^0 \cap \sigma^0. \)

Since \( g(0) = X \in T_{\sigma'} \subset T_\beta \) and \( Z = g(t_1) \in T_\eta \subset T_\beta, \) from Theorem 2.3 we know that the geodesic segment satisfies

\[
g(X, Z) \subset T_\beta.
\]

In particular, the initial tangents of \( g(Z, X) \) at \( Z \) vanishes in the component \( \mathbb{R}_{\geq 0}^\beta. \) Since \( \phi \) is reducible, it is not hard to see that \( k \geq 2; \) otherwise it contradicts the fact that the mapping class group acts properly on the Teichmüller space. Since the piecewise-geodesic \( g \) is the unique minimal length path in \( \overline{\text{Teich}(S)} \) joining \( g(0) \) to \( g(L) \) and intersecting the closures of the strata \( T_{\sigma_1}, T_{\sigma_2}, \ldots, T_{\sigma_k} \) in order, in particular the piecewise-geodesic \( g(X, Z) \cup g(Z, g(t_2)) \) is the minimal length path in \( \overline{\text{Teich}(S)} \) joining \( X \) and \( g(t_2) \) and intersecting the stratum \( T_{\eta}. \) From Theorem 2.14 we know that the initial tangents of \( g(Z, X) \) and \( g(Z, g(t_2)) \) are equal in the components \( \mathbb{R}_{\geq 0}^\beta. \) Equation (7) tells that the initial tangent of \( g(Z, X) \) at \( Z \) vanishes in the component \( \mathbb{R}_{\geq 0}^\beta. \) Then we have the initial tangents of \( g(Z, g(t_2)) \) at \( Z \) also vanishes in the component \( \mathbb{R}_{\geq 0}^\beta \) because \( \beta \in \eta^0. \) Thus, from Theorem 2.7 we know that the geodesics satisfies

\[
g(Z, g(t_2)) \subset T_\beta.
\]

In particular we have

\[
g(t_2) \in T_\beta.
\]

Similarly we use the same argument by induction on \( t_i, \) then we have

\[
g(t_i) \in T_\beta, \ \forall i = 1, 2, \ldots, k.
\]

In particular, we have

\[
g(t_k) \in T_\beta.
\]

On the other hand, from Part (1) we know that \( \sigma_i \subset \sigma \cup \sigma' \) for all \( i = 1, \ldots, k. \) Part (3) of Theorem (3.2) tells that

\[
\tau_{i,n} \in TW(\sigma_i), \ for \ i = 1, 2, \ldots, k - 1.
\]

So \( \tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \) is a product of Dehn-twists about the curves in \( \sigma \cup \sigma'. \) Since \( g_n(t_k) = \phi^n \circ Y, \) Part (3) of Theorem 3.2 tells that

\[
\lim_{n \to +\infty} \dist(\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ (\phi^n \circ Y), g(t_k)) = 0.
\]
Since $\phi^n$ is a product of Dehn-twists about the curves in $\sigma$, the mapping class $\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ \phi^n$ is also a product of Dehn-twists about the curves in $\sigma^0 \cup \sigma^n$. If we project both $\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ (\phi^n \circ Y)$ and $g(t_k)$ onto the moduli space of $S$, both $g(t_k)$ and $Y$ project onto the same point in the moduli space of $S$. Thus, there exists a mapping class $\phi' \in \text{Mod}(S)$ such that $Y = \phi' \circ g(t_k)$. Since $Y \in \text{Teich}(S)$, we also have

$$g(t_k) \in \text{Teich}(S)$$

which contradicts equation (12).

**Proof of Part (3).** For any $\epsilon > 0$, let $X' \in \text{Teich}(S)$ such that $\text{dist}(X', X) = \epsilon$. From Part (3) of Proposition 3.3 we know that, for each $\alpha_i \in \sigma^0$, there exists a point $X_{n,\epsilon}^i \in g(X', \phi^n \circ Y)$ such that

$$\lim_{n \to +\infty} \text{dist}(X_{n,\epsilon}^i, T_{\alpha_i}) = 0. \tag{13}$$

Since $\overline{\text{Teich}(S)}$ is a CAT(0) space, from Proposition 2.2 of Chapter II.2 in [BH99] we know that there exists $X_{n,\epsilon}^i \in g(X, \phi^n \circ Y)$ such that

$$\text{dist}(X_{n,\epsilon}^i, T_{\alpha_i}) \leq \max\{\text{dist}(X', X), \text{dist}(\phi^n \circ Y, \phi^n \circ Y)\}$$

$$= \epsilon.$$

Hence, the triangle inequality leads to

$$\text{dist}(X_{n,\epsilon}^i, T_{\alpha_i}) \leq \text{dist}(X_{n,\epsilon}^i, T_{\alpha_i}) + \text{dist}(X_{n,\epsilon}^i, X_{n,\epsilon}^i)$$

$$\leq \text{dist}(X_{n,\epsilon}^i, T_{\alpha_i}) + \epsilon$$

After taking a super limit, equation (13) tells that

$$\lim_{n \to +\infty} \sup \text{dist}(X_{n,\epsilon}^i, T_{\alpha_i}) \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, Part (3) follows from the inequality above and a standard diagonal argument.

We enclose this section by the following result, which is crucial to prove Theorem 1.3. Note that the following result requires to pass a subsequence, while Theorem 1.3 does not need to pass to a subsequence.

**Proposition 3.5.** Let $\sigma$ be a $m$-simplex and $\sigma^0 = \{\alpha_1, \cdots, \alpha_{m+1}\}$ and $\tau_i$ be a Dehn-twist about the curve $\alpha_i$ for $i = 1, 2, \cdots, m+1$. Let $\phi = \prod_{1 \leq i \leq m+1} \tau_i \in \text{Mod}(S)$ and $X, Y \in \text{Teich}(S)$, and $g_n$ be the geodesic $g(X, \phi^n \circ Y)$. Then, after passing to a subsequence, there exist a positive number $L$ and an associated partition $0 = t_0 < t_1 < \cdots < t_k = L$, and simplices $\sigma_0, \cdots, \sigma_k$ and a piecewise-geodesic

$$g : [0, L] \to \overline{\text{Teich}(S)}$$

with the following properties.

(1). $\sigma_i^0 \subset \sigma^0$, $\sigma_i^0 \cap \sigma_j^0 = \emptyset$ for $i \neq j$. Where $1 \leq i, j \leq m + 1$.

(2). $\sigma^0 = \bigcup_{i=1}^{m+1} \sigma_i^0$.

(3). $g(t_i) \in T_{\sigma_i}$, $i = 1, \cdots, k - 1$, $g(0) = X, g(t_k) = Y$.

(4). There are elements $\tau_i, n \in Tw(\sigma_i)$, for $i = 1, \cdots, k - 1$ so that $g_n[0, t_i]$ converges in $\overline{\text{Teich}(S)}$ to the restriction $g([0, t_i])$ and for each $i = 1, \cdots, k - 1$ and $t \in [t_i, t_{i+1}]$,

$$\lim_{n \to +\infty} \text{dist}(\tau_i, n \circ \cdots \circ \tau_1, n \circ (g_n(t)), g(t)) = 0.$$
In particular, \( \tau_{k-1,n} \circ \cdots \circ \tau_{1,n} = \phi^{-n} \) when \( n \) is big enough.

(5). The piecewise-geodesic \( g \) is the unique minimal length path in \( \overline{\Teich(S)} \) joining \( g(0) \) to \( g(L) \) and intersecting the closures of the strata \( T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}} \) in order.

**Proof.** Part (2) of Proposition (3.3) tells that after passing to a subsequence of \( \{g(X, \phi^n \circ Y) \} \), there exits a constant \( t_1 > 0 \) such that

(a). the sequence of geodesic segments \( \{g(X, \phi^n \circ Y)([0, t_1])\} \) converges to a geodesic \( g([0, t_1]) \) in \( \overline{\Teich(S)} \).

(b). There exists a simplex \( \sigma_1 \) such that \( g(t_1) \in T_{\sigma_1} \).

From Part (2) of Proposition 3.3 we know that

\[ \sigma_1^0 \subset \sigma_1. \]

Set

\[ \tau_{1,n} = \Pi_{\gamma \in \sigma_1^0} \tau_1^\gamma. \]

Part (3) of Theorem 3.2 tells that

\[ \lim_{n \to +\infty} \text{dist}(\tau_{1,n} \circ (g_n(t)), g(t)) = 0, \quad \forall t \in [0, t_1]. \]

Since \( g(t_1) \in T_{\sigma_1} \), we have

\[ g(g(t_1), \phi^n \circ Y) = \tau_{1,n} \circ g(g(t_1), \Pi_{\gamma \in (\sigma_0 - \sigma_1^0) \tau_1^\gamma \circ Y}). \]

Let \( \sigma_1' \) be a simplex with vertices \( \sigma_1^0 = \sigma_0 - \sigma_1^0 \). Since \( \sigma_1 \) is disjoint with \( \sigma_1' \), the geodesics \( \{g(g(t_1), \Pi_{\gamma \in (\sigma_0 - \sigma_1^0) \tau_1^\gamma \circ Y})\} \) satisfy the conditions of Proposition 3.4. Hence, by applying Proposition 3.4 on the geodesics \( \{g(g(t_1), \Pi_{\gamma \in (\sigma_0 - \sigma_1^0) \tau_1^\gamma \circ Y})\} \) then there exists a constant \( t_2 \) with \( t_2 > t_1 > 0 \) and a simplex \( \sigma_2 \) such that

(c). \( g(g(t_1), \Pi_{\gamma \in (\sigma_0 - \sigma_1^0) \tau_1^\gamma \circ Y}) \) converges to a geodesic \( g([t_1, t_2]) \) in \( \overline{\Teich(S)} \),

(d). \( g(t_2) \in T_{\sigma_2} \subset T_{\sigma_1} \) which in particular indicates that

\[ \sigma_2 \subset \sigma \text{ and } \sigma_1 \cap \sigma_2 = \emptyset. \]

Set

\[ \tau_{2,n} = \Pi_{\gamma \in \sigma_1^0} \tau_1^\gamma. \]

By Part (3) of Theorem 3.2 again we know that the piecewise-geodesics \( \{g(g(t_1), g(t_2)) \cup g(g(t_2), \Pi_{\gamma \in (\sigma_1^0 - \sigma_1^0) \tau_1^\gamma \circ Y})\} \) and the geodesics \( \{g(X, \Pi_{\gamma \in (\sigma_0 - \sigma_1^0) \tau_1^\gamma \circ Y})\} \) will coincide as \( n \to \infty \). Since the piecewise-geodesics \( \{g(X, g(t_1)) \cup g(g(t_1), \phi^n \circ Y)\} \) and the geodesics \( \{g(X, \phi^n \circ Y)\} \) will also coincide as \( n \to \infty \), we have

\[ \lim_{n \to +\infty} \text{dist}(\tau_{2,n} \circ \tau_{1,n} \circ (g_n(t)), g(t)) = 0, \quad \forall t \in [t_1, t_2]. \]

Since \( \sigma \) is a simplex of finite dimension, by induction we have, after finite times through applying Proposition 3.4, then there exists a sequence of simplexes \( \{\sigma_i\}_{i=1, \ldots, k} \) and a sequence of positive numbers \( \{t_i\}_{i=1, \ldots, k} \) such that

(1). \( \sigma_i^0 \subset \sigma_i \), \( \sigma_i^0 \cap \sigma_j^0 \) is empty for \( i \neq j \), where \( 1 \leq i, j \leq m + 1 \).

(2). \( \sigma^0 = \bigcup_{i=1}^k \sigma_i^0 \).

(3). \( g(t_i) \in T_{\sigma_i}, i = 1, \cdots, k, \) \( g(0) = X, g(t_k) = Y \).

(4). There are elements \( \tau_{i,n} \in Tw(\sigma_i) \), for \( i = 1, \cdots, k \) so that, \( g_n([0, t_1]) \) converges in \( \overline{\Teich(S)} \) to the restriction \( g([0, t_1]) \) and for each \( i = 1, \cdots, k \) and \( t \in [t_i, t_{i+1}] \),

\[ \lim_{n \to +\infty} \text{dist}(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ (g_n(t)), g(t)) = 0. \]
Since \( g(t_k) = Y \),
\[
\lim_{n \to \infty} \text{dist}(\tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ \phi^n \circ Y), Y) = 0.
\]
This only happens when \( \tau_{k-1,n} \circ \cdots \circ \tau_{1,n} \circ \phi^n \) is identity when \( n \) is large enough.

It follows from Part (5) of Theorem 3.2 that the piecewise-geodesic \( g \) is the unique minimal length path in \( \overline{\text{Teich}(S)} \) joining \( g(0) \) to \( g(L) \) and intersecting the closures of the strata \( T_{\sigma_1}, T_{\sigma_2}, \ldots, T_{\sigma_{k-1}} \) in order. So Part (5) also follows. \( \square \)

4. PROOFS OF THEOREM 1.2, 1.3 AND 1.4

In [Yam10] Yamada constructed the Teichmüller-Coxeter complex \( D(\overline{\text{Teich}(S)}, \iota) \) through introducing an infinite Coxeter reflection group and gluing infinite copies of \( \overline{\text{Teich}(S)} \) through the stratum. In this section, we use \( D(\overline{\text{Teich}(S)}, \iota) \) as a bridge to finish the proofs of Theorem 1.2 and 1.3.

First we briefly review the Teichmüller-Coxeter complex. One can refer to [Yam10, Wol10] for more details. For each simplex \( \sigma \), associate a formal reflection group \( W_\sigma \) with one reflection generator \( \omega_\sigma \) for each simple closed curve \( \alpha \in \sigma^0 \), with \( \omega_\alpha^2 = id \) and commuting generators. For an inclusion of simplices \( \sigma \subset \tau \) associate the natural injective homomorphism \( \psi_{\tau \sigma} : W_\sigma \to W_\tau \) satisfying
\[
\psi_{\tau \rho} = \psi_{\tau \sigma} \psi_{\sigma \rho} \text{ for } \rho \subset \sigma \subset \tau.
\]
The system of groups and monomorphisms \( \{ W_\sigma, \psi_{\tau \sigma} \} \) has a direct limit \( \hat{W} \) the Coxeter group of curves. The injectivity of the homomorphisms provides that the homomorphisms: \( i_\sigma : W_\sigma \to \hat{W} \) are injective. The Teichmüller-Coxeter development \( D(\overline{\text{Teich}(S)}, \iota) \) is the quotient of \( \hat{W} \times \overline{\text{Teich}(S)} \) by the equivalence relation
\[
(\omega, Y) \sim (\omega', Y') \text{ provided } Y = Y' \text{ and } \omega^{-1} \omega' \in W_\sigma(Y)
\]
where \( \sigma(Y) \) is the simplex of null lengths for the surface \( Y \).

The following theorem was proved in [Yam10].

**Theorem 4.1** (Yamada). (1). The complex \( D(\overline{\text{Teich}(S)}, \iota) \) is a complete CAT(0) space. In particular, for any two points \((\omega_1, y_1), (\omega_2, y_2) \in D(\overline{\text{Teich}(S)}, \iota)\) there exists a unique geodesic joining \((\omega_1, y_1) \) and \((\omega_2, y_2) \).

(2). Let \( \sigma \) be a simplex and \( Z \in T_\sigma \), the Alexandrov tangent cone of \( D(\overline{\text{Teich}(S)}, \iota) \) at \((1, Z)\) is the vector space \( \mathbb{R}^{[\sigma]} \times T_Z T_\sigma \).

Let \( \sigma \) be a simplex and \( Z \) be a point in \( T_\sigma \). Since \( \Pi_{\alpha \in (\sigma^0 \omega_\sigma, Z)} \), the definition of \( D(\overline{\text{Teich}(S)}, \iota) \) tells that
\[
(1, Z) = (\Pi_{\alpha \in \sigma^0 \omega_\sigma}, Z).
\]

The following lemma gives a new viewpoint for Proposition 2.14 in the setting of Teichmüller-Coxeter development.

**Lemma 4.2.** Given a simplex \( \sigma \) and let \((X, Y) \) be two points in \( \overline{\text{Teich}(S)} \). Let \( Z \in T_\sigma \) such that the piecewise-geodesic \( g(X, Z) \cup g(Z, Y) \) is the minimal length path in \( \overline{\text{Teich}(S)} \) joining \( X \) and \( Y \) and intersecting the stratum \( T_\sigma \). Then, the piecewise-geodesic \( g((1, X), (1, Z)) \cup g((\omega, Z), (\omega, Y)) \) is a global geodesic in the Teichmüller-Coxeter development \( D(\overline{\text{Teich}(S)}, \iota) \) where \( \omega = \Pi_{\alpha \in (\sigma^0 \omega_\sigma, Z)} \).
Proof. It suffices to show the piecewise-geodesic $g((1, X), (1, Z)) \cup g((\omega, Z), (\omega, Y))$ is smooth at $(1, Z) = (\omega, Z)$. Since the Alexandrov tangent cone of $D(\Teich(S), \iota)$ at $(1, Z)$ is a vector space, it is sufficient to show that the direction of the geodesic $g((1, Z), (1, X))$ at $(1, Z)$ is opposite to the direction of $g((\omega, Z), (\omega, Y))$ at $(\omega, Z)$. From Proposition 2.14 we know that the directions of the geodesics $g((Z, X))$ and $g(Z, Y)$ in $\Teich(S)$ at $Z$ satisfy

(a) The initial tangents of $g(Z, X)$ and $g(Z, Y)$ are equal in the components $\mathbb{R}^{[\sigma]}_{\geq 0}$.

(b) The sum of the initial tangents of $g(Z, X)$ and $g(Z, Y)$ at $Z$ has vanishing projection into the subcone $T_Z T_{\sigma}$.

From the construction of the Teichmüller-Coxeter development we know that (a) tells that

(c) The initial tangents of $g((\omega, Z), (\omega, Y))$ and $g((1, Z), (1, Y))$ are opposite in the components $\mathbb{R}^{[\sigma]}$ in Part (2) of Theorem 4.1.

And (b) tells that

(d) The sum of the initial tangents of $g((\omega, Z), (\omega, Y))$ and $g((1, Z), (1, Y))$ at $(1, Z)$ has vanishing projection into the subcone $T_Z T_{\sigma}$ in Part (2) of Theorem 4.1.

(c) and (d) exactly tell that the direction of the geodesic $g((1, Z), (1, X))$ at $(1, Z)$ is opposite to the direction of $g((\omega, Z), (\omega, Y))$ at $(\omega, Z)$. That is, the piecewise-geodesic $g((1, X), (1, Z)) \cup g((\omega, Z), (\omega, Y))$ is smooth at $(1, Z) = (\omega, Z)$.

The following result is a restatement of Proposition 3.5 in the setting of the Teichmüller-Coxeter development.

**Proposition 4.3.** Let $\sigma$ be a $m$-simplex and $\sigma^0 = \{\alpha_1, \ldots, \alpha_{m+1}\}$ and $\tau_i$ be a Dehn-twist about the curve $\alpha_i$ for $i = 1, 2, \ldots, k + 1$. Let $\phi = \prod_{1 \leq i \leq m+1} \tau_i \in \Mod(S)$ and $X, Y \in \Teich(S)$, and $g_n$ be the geodesic $g(X, \phi^n \circ Y)$. Then, for any piecewise-geodesic in the limits of $\{g(X, \phi^n \circ Y)\}$ in sense of Proposition 3.5, i.e., there exist a positive number $L$ and an associated partition $0 = t_0 < t_1 < \cdots < t_k = L$, and simplices $\sigma_0, \ldots, \sigma_k$ and a piecewise-geodesic $g : [0, L] \to \Teich(S)$, we have the piecewise-geodesic

$$\bigcup_{j=0}^{k-1} g((\prod_{i=0}^{j} \Pi_{\alpha_1 \in \sigma_1, \omega_1, g(t_j)}), (\prod_{i=0}^{j+1} \Pi_{\alpha_1 \in \sigma_1, \omega_1, g(t_{j+1})}))$$

is a global geodesic joining $(1, X)$ and $(\prod_{i=0}^{k-1} \Pi_{\alpha_1 \in \sigma_1, \omega_1, g(t_j+1)})$ in $D(\Teich(S), \iota)$ which passes through the points $(\prod_{i=0}^{k+1} \Pi_{\alpha_1 \in \sigma_1, \omega_1, g(t_{j+1})}), j = 1, \ldots, k - 1$.

**Proof.** Combining Part (5) of Proposition 3.5 and Lemma 4.2.

Now we are ready to show that the sequence of the directions of the geodesics $\{g(X, \phi^n \circ Y)\}$ is convergent in $V_X(S)$ as $n \to \infty$ if $\phi$ is a multi Dehn-twist.

**Theorem 4.4.** Let $\sigma$ be a $m$-simplex and $\sigma^0 = \{\alpha_1, \ldots, \alpha_{m+1}\}$ and $\tau_i$ be a Dehn-twist about the curve $\alpha_i$ for $i = 1, 2, \ldots, m+1$. Let $\phi = \prod_{1 \leq i \leq m+1} \tau_i \in \Mod(S)$ and $X, Y \in \Teich(S)$, and $g_n$ be the geodesic $g(X, \phi^n \circ Y)$. Then, there exist a positive number $t_1$ and a subsimplex $\sigma_1$ of $\sigma$, and a geodesic $g : [0, t_1] \to \Teich(S)$ such that $g(t_1) \in T_{\sigma_1}$ and $g\left((0, t_1]\right)$ converges to $g([0, t_1])$ as $n$ goes to infinity. In particular,
the sequence of the directions of the geodesics \( \{g(X, \phi^n \circ Y)\} \) is convergent in \( V_X(S) \) as \( n \to \infty \).

**Proof.** From Theorem 3.2 it is sufficient to show that the sequence of the directions of the geodesics \( \{g(X, \phi^n \circ Y)\} \) is convergent in \( V_X(S) \) as \( n \to \infty \).

We argue by contradiction. Assume not. Since the direction \( V_X(S) \) of \( \text{Teich}(S) \) at \( X \) is a sphere \( S^{6g-7+2n} \), which is compact, there exist two subsequences \( \{g(X, \phi^{n_k} \circ Y)\}_{k \geq 1} \) and \( \{g(X, \phi^{n_k^2} \circ Y)\}_{k \geq 1} \) of \( \{g(X, \phi^n \circ Y)\}_{n \geq 1} \) such that their limits in \( V_X(S) \) are different.

We first consider the sequence of geodesics \( \{g(X, \phi^{n_k} \circ Y)\}_{k \geq 1} \). From Proposition 3.5, after passing to a subsequence of \( \{n_k\}_{k \geq 1} \) (we still denote it by \( \{n_k\}_{k \geq 1} \)), there exist a positive number \( L_1 \) and an associated partition \( 0 = t_0 < t_1 < \cdots < t_k = L_1 \), and simplices \( \sigma_0, \cdots, \sigma_k \) and a piecewise-geodesic

\[
g^1 : [0, L] \to \text{Teich}(S)
\]

with the following properties.

1. \( \sigma_i^0 \subset \sigma^0, \sigma_i^0 \cap \sigma_j^0 = \emptyset \) for all \( 1 \leq i \neq j \leq k \).

2. \( \sigma^0 = \bigcup_{i=1}^k \sigma_i^0 \).

3. \( g^1(t_i) \in T_{\sigma_i}, i = 1, \cdots, k-1, g^1(0) = X, g^1(t_k) = Y \).

4. There are elements \( \tau_i, n \in Tw(\sigma_i) \), for \( i = 1, \cdots, k-1 \) so that \( g_n([0, t_1]) \) converges in \( \text{Teich}(S) \) to the restriction \( g^1([0, t_1]) \) and for each \( i = 1, \cdots, k-1 \) and \( t \in [t_i, t_{i+1}] \),

\[
\lim_{n \to +\infty} \text{dist}(\tau_{i,n} \circ \cdots \circ \tau_{1,n} \circ (g_n(t)), g^1(t)) = 0.
\]

In particular, \( \tau_{k-1,n} \circ \cdots \circ \tau_{1,n} = \phi^{-n} \) when \( n \) is big enough.

5. The piecewise-geodesic \( g^1 \) is the unique minimal length path in \( \text{Teich}(S) \) joining \( g^1(0) \) to \( g^1(L_1) \) and intersecting the closures of the strata \( T_{\sigma_1}, T_{\sigma_2}, \cdots, T_{\sigma_{k-1}} \) in order.

Since \( \sigma_i^0 \cap \sigma_j^0 = \emptyset \) for all \( 1 \leq i \neq j \leq k \) and \( \sigma^0 = \bigcup_{i=1}^k \sigma_i^0 \),

\[
\prod_{j=0}^{k-1} \left( \prod_{\gamma \in \sigma_i^0} \omega_\gamma \right) = \prod_{\alpha \in \sigma^0} \omega_\alpha.
\]

From Proposition 4.3 and the equation (14) there exist positive numbers \( t_1, L_1 \) and a geodesic \( l_1 : [0, L_1] \to D(\text{Teich}(S), \iota) \) such that

(a). \( l_1(0) = (1, X), l_1(L_1) = (\prod_{\alpha \in \sigma^0} \omega_\alpha, Y) \).

(b). The first point where the geodesic \( l_1 \) meets the wall of \( D(\text{Teich}(S), \iota) \) (the singular set) is \( g^1(t_1) = l_1(t_1) \), where we identify \((1, \text{Teich}(S)) \) with \( \text{Teich}(S) \).

Similarly we make the same argument as above for the geodesics \( \{g(X, \phi^{n_k^2} \circ Y)\}_{k \geq 1} \) (we still denote it by \( \{n_k^2\}_{k \geq 1} \)), there exist positive numbers \( s_1, L_2 \) and a subsimplex \( \beta_1 \) of \( \sigma \), and a geodesic \( l_2 : [0, L_2] \to D(\text{Teich}(S), \iota) \) satisfying

(c). \( l_2(0) = (1, X), l_2(L_2) = (\prod_{\alpha \in \sigma^0} \omega_\alpha, Y) \).

(d). The first point where the geodesic \( l_2 \) meets the wall of \( D(\text{Teich}(S), \iota) \) (the singular set) is \( l_2(s_1) \).

Since the limit of the directions of the geodesics \( \{g(X, \phi^{n_k} \circ Y)\}_{k \geq 1} \) and \( \{g(X, \phi^{n_k^2} \circ Y)\}_{k \geq 1} \) is different in \( V_X(S) \), we have \( l_1(t_1) \neq l_2(s_1) \).
On the other hand, both $l_1$ and $l_2$ are geodesics joining $(1, X)$ and $(\prod_{\alpha \in \sigma \cup \omega} \omega_{\alpha}, Y)$. From Part (1) of Theorem 4.1 we know that the two geodesics $l_1$ and $l_2$ coincide, which contradicts our assumption that $\{g(X, \phi^{n_1} \circ Y)\}_{k \geq 1}$ and $\{g(X, \phi^{n_2} \circ Y)\}_{k \geq 1}$ have different limits in $V_X(S)$. \hfill $\square$

**Remark 4.1.** The statement in Theorem 4.4 is still true if we allow $X$ belong to a stratum $T_\eta$ where $\eta$ is disjoint with $\sigma$ by using the same argument as above.

Now we are ready to prove Theorem 1.2, 1.3 and 1.4.

**Proof of Theorem 1.2.** From Theorem 2.4 we know that $\phi$ is one of the four cases: irreducible with either $L_{WP}(\phi) = 0$ or $L_{WP}(\phi) > 0$ and reducible with either $L_{WP}(\phi) = 0$ or $L_{WP}(\phi) > 0$. We prove it through these four different types.

**Case (I).** $\phi$ is irreducible and $\delta = L_{WP}(\phi) = 0$.

From Theorem 2.4 we know that there exists an integer $k$ with $\phi^k = id$. Hence, the sequence of geodesics $\{g(X, \phi^{kn} \circ Y)\}$ is just the fixed geodesic $g(X, Y)$. So the limit of the sequence of the directions of the geodesics $g(X, \phi^{kn} \circ Y)$ is the direction of $g(X, Y)$.

**Case (II).** $\phi$ is irreducible and $\delta = L_{WP}(\phi) > 0$

From Theorem 2.4, there exists a unique bi-infinite Weil-Petersson geodesic $\gamma$ in $\text{Teich}(S)$ such that $\phi \circ \gamma(t) = \gamma(t + \delta)$ for all $t \in \mathbb{R}$. Consider the geodesic triangles $\{\Delta(X, \phi^n \circ Y, \phi^n \circ \gamma(0))\}$ in $\text{Teich}(S)$ with vertices $\{X, \phi^n \circ Y, \phi^n \circ \gamma(0)\}$. Since $\text{Mod}(S)$ acts on $\text{Teich}(S)$ by isometries, we have for all $n \geq 1$,

$$\text{dist}(\phi^n \circ Y, \phi^n \circ \gamma(0)) = \text{dist}(Y, \gamma(0)).$$

On the other hand, the triangle inequality tells that for all $n \geq 1$ we have

$$\text{dist}(X, \phi^n \circ \gamma(0)) \geq \text{dist}(\gamma(0), \phi^n \circ \gamma(0)) - \text{dist}(X, \gamma(0)) = n \cdot \delta - \text{dist}(X, \gamma(0)).$$

Hence, $\text{dist}(X, \phi^n \circ \gamma(0))$ goes to $+\infty$ as $n \to +\infty$. From the standard argument in CAT(0) geometry (see the proof of Proposition 8.2 in [BH99]) we know that the limit of the sequence of the directions of $\{g(X, \phi^n \circ \gamma(0))\}$ exists in visual sphere at $X$. Since $\{\text{dist}(\phi^n(Y), \phi^n \circ \gamma(0))\}_{n \geq 1}$ is uniformly bounded and $\lim_{n \to \infty} \text{dist}(X, \phi^n \circ \gamma(0)) = +\infty$, standard CAT(0) geometry (see the proof of Proposition 8.2 in [BH99]) tells that the limit of the sequence of the directions of $\{g(X, \phi^n \circ Y)\}$ exists and the limit is the same as the limit of the sequence of the directions of $\{g(X, \phi^n \circ \gamma(0))\}$ in $V_X(S)$.

**Case (III).** $\phi$ is reducible and $\delta = L_{WP}(\phi) > 0$

We follows the same idea as in the proof of Case (II). From Theorem 2.4, there exist an integer $k$ and a bi-infinite Weil-Petersson geodesic $\gamma(t) \subset \text{Teich}(S)$ such that for all $t \in \mathbb{R}$ and $k \in \mathbb{Z}$, $\phi^k \circ \gamma(t) = \gamma(t + k\delta)$. We consider the geodesic triangles $\{\Delta(X, \phi^{kn} \circ Y, \phi^{kn} \circ \gamma(0))\}$ in $\text{Teich}(S)$ with vertices $\{X, \phi^{kn} \circ Y, \phi^{kn} \circ \gamma(0)\}$. Use totally the same argument as in Case II we get the limit of the sequence of the directions of $\{g(X, \phi^{kn} \circ Y)\}$ exists and the limit is the same as the limit of the sequence of the directions of $\{g(X, \phi^{kn} \circ \gamma(0))\}$ in $V_X(S)$.

**Case (IV).** $\phi$ is reducible and $\delta = L_{WP}(\phi) = 0$


From Proposition 2.5 we know that there exists an integer $k$ such that $\phi^k$ is a multi Dehn-twist. Then the conclusion directly follows from Theorem 4.4. \qed

**Proof of Theorem 1.3.** First it follows from Theorem 4.4 that there exist a positive number $t_1$ and a subsimplex $\sigma_1$ of $\sigma$, and a geodesic $g : [0, t_1) \to \text{Teich}(\Sigma)$ such that $g(t_1) \in T_{\sigma_1}$ and $g_n([0, t_1])$ converges to $g([0, t_1])$ as $n$ goes to infinity.

Consider the geodesics $\{g(g(t_1), \prod_{\alpha \in \sigma^0 - \sigma^0_1} \tau^n_\alpha \circ Y)\}$, it follows from Remark 4.1 that there exist a positive number $t'_2$ and a subsimplex $\sigma_2$ of $\sigma$, and a geodesic $g' : [0, t'_2] \to \text{Teich}(\Sigma)$ such that

(a). $\sigma_2 \cap \sigma_1 = \emptyset$.

(b). $g'(0) = g(t_1)$, $g'(t'_2) \in T_{\sigma_2}$ and the geodesic segments $\{g(g(t_1), \prod_{\alpha \in \sigma^0 - \sigma^0_1} \tau^n_\alpha \circ Y)\}([0, t'_2])$ converge to the geodesic $g'([0, t'_2])$ as $n$ goes to infinity.

Denote $\prod_{\alpha \in \sigma^0_1} \tau^n_\alpha$ by $\tau_{1,n}$. Since $g(g(t_1), \phi^n \circ Y) = \prod_{\alpha \in \sigma^0_1} \tau^n_\alpha \circ Y = \prod_{\alpha \in \sigma^0 - \sigma^0_1} \tau^n_\alpha \circ Y$, we have

$$\lim_{n \to +\infty} \text{dist}(\tau_{1,n} \circ g_n(t) g(t), g(t')) = 0, \quad \forall t \in [0, t'_2].$$

Setting $t_2 = t_1 + t'_2$. We extend $g$ be to defined on $[0, t_2]$ as follows

$$g(t) = \begin{cases} g(t), & \forall t \in [0, t_1] \\ g'(t - t_1), & \forall t \in [t_1, t_2] \end{cases}.$$ 

Then we get a piecewise-geodesic $g(X, g(t_1)) \cup g(g(t_1), g(t_2))$ satisfying

(c). $\sigma^0_1 \subset \sigma^0$ for $i = 1, 2$.

(d). $g(t_1) \in T_{\sigma_1}$, $i = 1, 2, g(0) = X$.

(e). $g_n([0, t_1])$ converges in $\text{Teich}(\Sigma)$ to the restriction $g([0, t_1])$, and

$$\lim_{n \to +\infty} \text{dist}(\tau_{1,n} \circ g_n(t), g(t)) = 0, \quad \text{for} \quad t \in [t_1, t_2].$$

Where $\tau_{1,n} = \prod_{\alpha \in \sigma_1} \tau^{-n}_\alpha$.

If we only consider the geodesics $\{g(X, \phi^n \circ Y)[0, t_2]\}$, it follows from Theorem 3.2 that the piecewise-geodesic $g(X)$ is the unique minimal length path in $\text{Teich}(\Sigma)$ joining $g(0)$ to $g(t_2)$ and intersecting the closures of the stratum $T_{\sigma_1}$.

Since the cardinality $|\sigma| = k < +\infty$, after applying the arguments above, by induction we know that it will end in finite steps.

Then, Part (1), (2), (3), (4) and (5) of the conclusion follow from Proposition 3.5 and the argument above. And Part (6) directly follows from Proposition 4.3. \qed

**Proof of Theorem 1.4.** We argue by contradiction. Assume not. Then there exist two subsequences $\{k^1_n\}_{n \geq 1}$ and $\{k^2_n\}_{n \geq 1}$ of $\{n\}_{n \geq 1}$ such that

$$\lim_{n \to +\infty} \ell(g(X, \phi^{k^1_n} \circ Y)) \neq \lim_{n \to +\infty} \ell(g(X, \phi^{k^2_n} \circ Y)).\tag{15}$$

From Theorem 1.3, the geodesics $\{g(X, \phi^{k^1_n} \circ Y)\}_{n \geq 1}$ induces a geodesic $c$ in $D(\text{Teich}(\Sigma), t)$ joining $(1, X)$ and $(\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$. Part (4) of Theorem 1.3 tells that

$$\lim_{n \to +\infty} \ell(g(X, \phi^{k^1_n} \circ Y)) = \ell(c).\tag{16}$$

Similarly, from Theorem 1.3 we know that $\{g(X, \phi^{k^2_n} \circ Y)\}$ also induces a geodesic $d$ in $D(\text{Teich}(\Sigma), t)$ joining $(1, X)$ and $(\prod_{\alpha \in \sigma^0} \omega_\alpha, Y)$. Part (4) of Theorem 1.3 tells that
that
\[ \lim_{n \to +\infty} \ell(g(X, \phi_{k_n} \circ Y)) = \ell(d). \]

Since both \(c\) and \(d\) are geodesics connecting \((1, X)\) and \((\prod_{\alpha \in \sigma^0} \omega_{\alpha}, Y)\), Part (1) of Theorem 4.1 tells that \(c\) coincide \(d\). In particular, we have
\[ \ell(c) = \ell(d). \]

(18)

It is clear that equations (15), (16), (17) and (18) can not hold at the same time. Hence, we get a contradiction.

\[ \square \]

5. PROOFS OF THEOREM 1.5 AND PROPOSITION 1.6

Let \(M\) be a complete CAT(0) space. Recall two geodesic rays in a complete CAT(0) space are asymptotic if their Hausdorff distance is finite. We call two geodesic rays \(c_i (i = 1, 2) : [0, +\infty) \to M\) are strongly asymptotic if
\[ \lim_{t \to \infty} \text{dist}_M(c_1(t), c_2([0, \infty))) = 0 \]
where \( \text{dist}_M\) is the distance function on \(M\).

In [BMM10] the authors applied the Gauss-Bonnet formula and ruled surface method to show that two asymptotic geodesic rays in \(\text{Teich}(S)\) are strongly asymptotic if one of them is recurrent. Where a geodesic ray \(c : [0, +\infty) \to \text{Teich}(S)\) is called recurrent provided that there exist a positive constant \(\epsilon\) and a sequence \(\{t_n\} \geq 1\) with \(t_n \to +\infty\) as \(n \to \infty\) such that \(c(t_n) \in \text{Teich}(S)_{\geq \epsilon}\) where
\[ \text{Teich}(S)_{\geq \epsilon} = \{ X \in \text{Teich}(S); \ell_\alpha(X) \geq \epsilon, \ \forall \text{ essential simple closed curve } \alpha \subset S \}. \]

The following result allows the initial point lie in the boundary \(\partial \text{Teich}(S)\), which will be used in next section.

**Proposition 5.1** (Brock-Masur-Minsky). Let \(c : [0, +\infty) \to \text{Teich}(S)\) be a recurrent ray and \(X \in \overline{\text{Teich}(S)}\). Then the unique geodesic ray emanating from \(X\) with finite Hausdorff distance with \(c([0, \infty))\) \(c' : [0, +\infty) \to \overline{\text{Teich}(S)}\) is strongly asymptotic to \(c(\mathbb{R}^{\geq 0})\).

**Proof.** Let \(d : [0, \text{dist}(X, c(0))] \to \overline{\text{Teich}(S)}\) be the geodesic joining \(X\) and \(c(0)\) with unit speed. Since \(c(0) \in \text{Teich}(S)\), from Theorem 2.3 we know that
\[ d(0, \text{dist}(X, c(0))) \subset \text{Teich}(S). \]
We argue by contradiction. Assume not. That is there exists a constant \(\delta > 0\) such that
\[ \inf_{t \geq 0} \text{dist}(c(t), c'(\mathbb{R}^{\geq 0})) \geq \delta. \]

Choose the point \(d(\frac{\delta}{2})\) and consider the geodesic ray emanating from \(d(\frac{\delta}{2})\) with finite Hausdorff distance with \(c(\mathbb{R}^{\geq 0})\), which is denoted by \(c'' : [0, +\infty) \to \text{Teich}(S)\). Since \(d(\frac{\delta}{2}) \in \text{Teich}(S)\) and \(c([0, \infty))\) is recurrent, Theorem 4.1 in [BMM10] tells that
\[ \lim_{t \to \infty} \text{dist}(c(t), c''([0, \infty))) = 0. \]

(20)
Since the distance function between two convex subsets in a complete CAT(0) space is convex (see [BH99]), we have

\[
\sup_{t \geq 0} \text{dist}(c'(t), c''([0, \infty])) \leq \text{dist}(c''(0), c'(0)) = \frac{\delta}{2}.
\]

It is clear that inequalities (19), (21) and equation (20) can not hold at the same time. Then we get a contradiction. \(\square\)

**Proposition 5.2.** Let \(c : [0, +\infty) \to \text{Teich}(S)\) be a recurrent geodesic ray. Then, for any essential simple closed curve \(\alpha \subset S\) we have

\[
\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty.
\]

**Proof.** First by Theorem 2.1 we know that \(\lim_{t \to +\infty} \ell_\alpha(c(t))\) is either infinite or finite. We argue by contradiction. Assume not. Theorem 2.1 tells that there exists a number \(C \geq 0\) such that

\[
\lim_{t \to +\infty} \ell_\alpha(c(t)) = C \geq 0.
\]

Therefore, the second derivative of \(\ell_\alpha\) satisfies

\[
\lim_{t \to +\infty} \ell''_\alpha(c(t)) = 0.
\]

On the other hand, since the geodesic \(c([0, \infty))\) is recurrent, there exist a constant \(\epsilon > 0\) and a sequence \(\{t_n\}_{n \geq 1}\) with \(t_n \to \infty\) as \(n \to \infty\) such that \(c(t_n) \in \text{Teich}(S)_{\geq \epsilon}\).

Theorem 3.11 in [Wol08] tells that there exists a constant \(c > 0\) such that

\[
\ell''_\alpha(c(t_n)) \geq c \cdot \ell_\alpha(c(t_n)), \quad \forall n \geq 1.
\]

In particular, we have

\[
\ell''_\alpha(c(t_n)) \geq c \cdot \epsilon > 0, \quad \forall n \geq 1
\]

which contradicts equation (22). \(\square\)

Now we are ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** Since \(\phi\) is pseudo-Anosov, from Theorem 2.4 we know that there exists a bi-infinite Weil-Petersson geodesic line \(r : (-\infty, +\infty) \to \text{Teich}(S)\) such that for all \(n \in \mathbb{Z}\)

\[
\phi^n \circ r(0) = r(n \cdot LWP(\phi)).
\]

In particular we have \(r(\mathbb{R})\) is recurrent. We assume that the limit geodesic ray of the geodesics \(\{g(X, r(n \cdot LWP(\phi)))\}\) is a geodesic ray \(c : [0, +\infty) \to \text{Teich}(S)\) satisfying \(c(0) = X\) and \(c([0, +\infty))\) is asymptotic to the geodesic ray \(r([0, +\infty))\) (see [BH99]). Since \(\text{Mod}(S)\) acts on \(\text{Teich}(S)\) by isometries, we have for all \(n \geq 1\),

\[
\text{dist}(\phi^n \circ Y, r(n \cdot LWP(\phi))) = \text{dist}(\phi^n \circ Y, \phi^n \circ r(0)) = \text{dist}(Y, r(0)) < +\infty.
\]

Hence, the two sequences of geodesics \(\{g(X, \phi^n \circ Y)\}\) and \(\{g(X, r(n \cdot LWP(\phi)))\}\) have the same limit geodesic ray \(c : [0, +\infty) \to \text{Teich}(S)\) which is asymptotic to the geodesic ray \(r([0, +\infty))\) (see [BH99]). Since \(r(t)\) is recurrent, from Theorem 4.1 in [BMM10] we have \(c : [0, +\infty) \to \text{Teich}(S)\) is strongly asymptotic to \(r([0, +\infty))\).
This implies that the geodesic ray $c(t)$ is also recurrent. Proposition 5.2 tells that for any essential simple closed curve $\alpha \subset S$, we have

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty.$$ 

□

It is interesting to know the rate of the growth in Theorem 1.5. Restricted on an axis of a pseudo-Anosov mapping class, Proposition 1.6 may give an answer.

Proof of Proposition 1.6. Let $\phi \in \text{Mod}(S)$ be pseudo-Anosov and $\gamma : \mathbb{R} \to \text{Teich}(S)$ be the unique axis of $\gamma$ (See [DW03]). Then, for all $n \in \mathbb{Z}$ we have

$$\phi^n \circ \gamma(0) = \gamma(n \cdot LWP(\phi)).$$

Let $\alpha$ be an essential simple closed curve on $S$. Since $\gamma(\mathbb{R})$ is recurrent, Theorem 2.1 and Proposition 5.2 tell that $\ell_\alpha(\gamma(t))$ is increasing after $t = t_0$ for some constant $t_0 \in \mathbb{R}$. In particular, for $n \in \mathbb{Z}^+$ large enough and $t \in [n \cdot LWP(\phi), (n+1) \cdot LWP(\phi)]$ we have

$$\frac{\ln \ell_\alpha(\gamma(n \cdot LWP(\phi)))}{(n+1) \cdot LWP(\phi)} \leq \frac{\ln \ell_\alpha(\gamma(t))}{t} \leq \frac{\ln \ell_\alpha(\phi^{n+1} \circ \gamma(0))}{nLWP(\phi)}.$$

Rewrite it as

$$\frac{\ln \ell_\alpha(\phi^n \circ \gamma(0))}{(n+1)LWP(\phi)} \leq \frac{\ln \ell_\alpha(\gamma(t))}{t} \leq \frac{\ln \ell_\alpha(\phi^{n+1} \circ \gamma(0))}{nLWP(\phi)}.$$

Since $\phi \in \text{Mod}(S)$ acts on $\text{Teich}(S)$ as an isometry, for all $n \geq 1$ we have

$$\ell_\alpha(\phi^n \circ \gamma(0)) = \ell_{\phi^{n}\circ \alpha}(\gamma(0)).$$

Inequality (23) and equation (24) tell that

$$\frac{\ln \ell_\alpha(\phi^n \circ \gamma(0))}{(n+1)LWP(\phi)} \leq \frac{\ln \ell_\alpha(\gamma(t))}{t} \leq \frac{\ln \ell_\alpha(\phi^{n+1} \circ \gamma(0))}{nLWP(\phi)}.$$

It is known that (see Theorem 14.23 in [FM12])

$$\lim_{n \to \infty} \frac{\ln \ell_\alpha(\phi^n \circ \gamma(0))}{n} = L_T(\phi).$$

Let $t \to \infty$ in inequality (25), Equation (26) gives that

$$\lim_{t \to \infty} \frac{\ln \ell_\alpha(\gamma(t))}{t} = \frac{L_T(\phi)}{LWP(\phi)}.$$

It follows from the Cauchy-Schwartz inequality that

$$\frac{L_T(\phi)}{LWP(\phi)} \geq \frac{1}{\sqrt{4\pi(g-1)}}$$

which is due to Linch in [Lin71]. □
6. Proof of Theorem 1.7

In this section we will study the limit geodesic ray in Theorem 1.2 when \( \phi \) is a reducible mapping class with positive translation length. Recall that Thurston's classification theorem (see [FLP12]) tells that each reducible mapping class \( \phi \in \text{Mod}(S) \) determines a maximal collection of essential simple closed curves \( \{ \alpha_i \} \) and a maximal collection of proper subsurfaces \( \{ PS_j \} \) of \( S \) such that there exists a positive integer \( k \) such that

\[
\phi^k = \left( \prod_i \tau_{\alpha_i} \right) \cdot \left( \prod_j \phi_j \right)
\]

where \( \tau_i \) is a Dehn-twist about \( \alpha_i \) and \( \phi_j = \phi^k|_{PS_j} \) is pseudo-Anosov on \( PS_j \).

Since the translation length of a multi Dehn-twist is zero, the pseudo-Anosov part is not empty if \( L_{WP}(\phi) > 0 \) (see Theorem 2.4). Let \( \sigma \) be a simplex such that \( \sigma^0 = (\cup_i \alpha_i) \cup (\cup_j \cup_{\beta \in \partial(PS_j)} \beta) \) where \( \partial(PS_j) \) is the boundary of \( PS_j \). The stratum \( T_\sigma \) is a product of low dimensional Teichmüller spaces \( \prod T' \times \prod_j T''_j \) with \( \phi^k \) fixing the factors, acting by a product of : the identity on \( T' \) and pseudo-Anosov elements \( \phi_j \) on \( T''_j \) with axis \( c_j \).

**Remark 6.1.** Let \( \phi \in \text{Mod}(S) \) be a reducible mapping class with \( L_{WP}(\phi) > 0 \) and \( c : (-\infty, +\infty) \to \text{Teich}(S) \) be an axis for \( \phi^k \). Theorem 2.4 tells that the projection of \( c(\mathbb{R}) \) onto a factor \( T''_j \) of \( T_\sigma \) is a geodesic line which is the axis for \( \phi_j \) on \( T''_j \).

From Proposition 1.6 we know that for any non-peripheral essential simple closed curve \( \alpha \) on \( PS_j \), we have \( \lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty \).

First let us consider the case that \( \phi \) does not have any Dehn-twist part. That is, there exist a positive integer \( k \) and a maximal collection of proper subsurfaces \( \{ PS_j \} \) of \( S \) such that \( \phi^k = \prod_j \phi_j \) where \( \phi_j = \phi^k|_{PS_j} \) is pseudo-Anosov on \( PS_j \).

Before we prove Theorem 1.7 let us prove some weaker statements.

**Proposition 6.1.** Let \( \phi \in \text{Mod}(S) \) be reducible and \( k \) be a positive number such that \( \phi^k = \prod_j \phi_j \) where \( \phi_j = \phi^k|_{PS_j} \) are pseudo-Anosov on \( PS_j \). Then for any \( X, Y \in \text{Teich}(S) \), the geodesics \( \{ g(X, \phi^kn \circ Y) \} \) converge to a geodesic ray \( c([0, +\infty)) \subset \text{Teich}(S) \) with \( c(0) = X \) as \( n \to +\infty \). Moreover, we have

1. For any non-peripheral essential simple closed curve \( \alpha \subset PS_j \), we have

\[
\lim_{t \to +\infty} \ell_\alpha(c(t)) = +\infty.
\]

2. There exists a positive number \( \epsilon_0 \) such that for any non-peripheral essential simple closed curve \( \beta \) in the complement \( (S - \cup_j PS_j) \) of \( \cup_j PS_j \) we have

\[
\epsilon_0 \leq \inf_{t \geq 0} \ell_\beta(c(t)) \leq \sup_{t \geq 0} \ell_\beta(c(t)) \leq \frac{1}{\epsilon_0}.
\]

3. There exists a positive number \( \epsilon_1 \) such that for any non-peripheral essential simple closed curve \( \gamma \) which intersects with at least one of simple closed curves \( \cup_j \cup_{\alpha \in \partial(PS_j)} \alpha \) we have

\[
\inf_{t \geq 0} \ell_\gamma(c(t)) \geq \epsilon_1.
\]

**Proof.** Let \( \sigma \) be a simplex with

\[
\sigma^0 = \cup_j (\cup_{\beta \in \partial(PS_j)} \beta).
\]
We let \( r : (-\infty, +\infty) \to T_0 \) be the axis for \( \phi^k \). That is, \( \phi^k \circ r(t) = r(t+kLWP(\phi)) \) for all \( t \in \mathbb{R} \) (since \( \overline{\text{Teich}(S)} \) is a complete CAT(0) space we have \( \text{LWP}(\phi^k) = k\text{LWP}(\phi) \)). Since \( \text{Mod}(S) \) acts on \( \overline{\text{Teich}(S)} \) by isometries, we have for all \( n \geq 1 \),

\[
\begin{align*}
\text{dist}(\phi^{kn} \circ Y, r(knLWP(\phi))) &= \text{dist}(\phi^{kn} \circ Y, \phi^{kn} \circ r(0)) \\
&= \text{dist}(Y, r(0)) \\
&< +\infty.
\end{align*}
\]

Thus, the limit geodesic ray of the geodesics \( \{g(X, \phi^{kn} \circ Y)\} \) is the same as the limit geodesic ray of the geodesics \( \{g(X, r(knLWP(\phi)))\} \) as \( n \to +\infty \), which is the unique ray emanating from \( X \) with finite Hausdorff distance to the geodesic ray \( r(\mathbb{R}^{>0}) \) (see the proof of Proposition 8.2 in \cite{BH99}). We denote this limit ray by \( c([0, +\infty)) \). From Theorem 2.3 we know that

\[
c([0, +\infty)) \subset \text{Teich}(S).
\]

Proof of Part (1). We argue by contradiction. Assume not. Then there exists a non-peripheral essential simple closed curve \( \alpha \subset PS_j \) for some \( j \) such that

\[
\lim_{t \to +\infty} \ell_\alpha(c(t)) \neq +\infty.
\]

Theorem 2.1 tells that there exists a number \( C_1 \geq 0 \) such that \( \ell_\alpha \) is decreasing along \( c(\mathbb{R}^{\geq 0}) \) and

\[
\lim_{t \to +\infty} \ell_\alpha(c(t)) = C_1 < +\infty.
\]

From Theorem 2.9 we know that there exists a positive number \( C_2 \) such that for all \( t \geq 0 \),

\[
\text{dist}(c(t), T_\alpha) \leq C_2.
\]

Let \( \{Z_i\}_{i \geq 1} \) be a sequence of points in \( T_\alpha \) with \( \text{dist}(c(i), Z_i) \leq C_2 + 1 \) for all \( i \geq 1 \). Since \( c(\mathbb{R}^{\geq 0}) \) and \( r(\mathbb{R}^{\geq 0}) \) are asymptotic, the triangle inequality tells that there exists a positive number \( C_3 \) such that for all \( i \geq 1 \) we have

\[
\text{dist}(r(i), Z_i) \leq C_3.
\]

Hence, the geodesics \( \{g(r(0), Z_i)\}_{i \geq 1} \) converge to the geodesic ray \( r(\mathbb{R}^{\geq 0}) \) (see the proof of Proposition 8.2 in \cite{BH99}). From Theorem 2.1 we know that for all \( i \geq 1 \),

\[
\max_{Z \in g(r(0), Z_i)} \ell_\alpha(Z) \leq \max\{\ell_\alpha(r(0)), \ell_\alpha(Z_i)\}
\]

\[
= \ell_\alpha(r(0)).
\]

Since \( \ell_\alpha \) is continuous on \( \overline{\text{Teich}(S)} \), taking a limit on inequality (29) as \( i \) goes to infinity, we have for all \( t \geq 0 \),

\[
\ell_\alpha(r(t)) \leq \ell_\alpha(r(0)) < \infty.
\]

On the other hand, from Remark 6.1 we have

\[
\ell_\alpha(r(t)) = +\infty
\]

which is a contradiction.

Proof of Part (2). For any non-peripheral essential simple closed curve \( \beta \) in the complement \((S - \cup_j PS_j)\) of \( \cup_j PS_j \), there exists another non-peripheral essential
simple closed curve $\beta'$ in the complement $(S - \cup_j PS_j)$ of $\cup_j PS_j$ such that $\beta'$ intersects with $\beta$.

Since $\phi^k$ acts as identity on $(S - \cup_j PS_j)$, we have for all $t \geq 0$,
\[
\ell_{\beta'}(r(t)) = \ell_{\beta'}(r(0)).
\]

Theorem 2.1 tells that for all $i \geq 1$ we have
\[
\max_{Z \in g(X,r(i))} \ell_{\beta'}(Z) \leq \max\{\ell_{\beta'}(X), \ell_{\beta'}(r(i))\} = \max\{\ell_{\beta'}(X), \ell_{\beta'}(r(0))\}.
\]

Since $c(\mathbb{R}^{\geq 0})$ is the limit ray of the geodesics $\{g(X, r(i))\}$ as $i \to +\infty$, after taking a limit as $i \to \infty$, we have
\[
(30) \quad \sup_{Z \in c(\mathbb{R}^{\geq 0})} \ell_{\beta'}(Z) \leq \max\{\ell_{\beta'}(X), \ell_{\beta'}(r(0))\} < \infty.
\]

Similarly we have
\[
(31) \quad \sup_{Z \in c(\mathbb{R}^{\geq 0})} \ell_{\beta}(Z) \leq \max\{\ell_{\beta}(X), \ell_{\beta}(r(0))\} < \infty.
\]

First the right hand side inequality of Part (2) follows from inequality (37).

Since $\beta$ intersects with $\beta'$, the left hand side inequality of Part (2) follows from inequality (30) and the Collar Lemma (see [Bus10]).

Proof of Part (3). Let $\alpha_0 \in \cup_j \cup_{\alpha \in \partial(PS_j)} \alpha$ such that $\gamma$ intersects with $\alpha_0$. Since $\ell_{\alpha_0}(r(i)) = 0$, from Theorem 2.1 we have for all $i \geq 1$,
\[
\max_{Z \in g(X,r(i))} \ell_{\alpha_0}(Z) \leq \max\{\ell_{\alpha_0}(X), \ell_{\alpha_0}(r(i))\} = \ell_{\alpha_0}(X).
\]

Since $c(\mathbb{R}^{\geq 0})$ is the limit geodesic ray of the geodesics $\{g(X, r(i))\}$ as $i \to +\infty$, after taking a limit as $i \to \infty$, we have
\[
(32) \quad \sup_{Z \in c(\mathbb{R}^{\geq 0})} \ell_{\alpha_0}(Z) \leq \ell_{\alpha_0}(X) < \infty.
\]

Then Part (3) follows from inequality (32) and the Collar Lemma (see [Bus10]).

The following lemma is elementary in CAT(0) geometry.

**Lemma 6.2.** Let $M$ be a complete CAT(0) space and $r_1, r_2 : [0, 1] \to M$ be two different geodesics. If $\text{dist}_M(r_1(t), r_2(t))$ is a positive constant for all $t \in [0, 1]$ and $\text{dist}(c_1(0), c_2([0, 1])) > 0$, then the convex hull of $r_1([0, 1]) \cup r_2([0, 1])$ in $M$ is isometric to a flat parallelogram in the two-dimensional Euclidean space $\mathbb{R}^2$.

**Proof.** The proof is the same as the proof of Theorem 2.11 in [BH99]. We leave it as an exercise for readers.

Before proving Theorem 1.7, we prove the following version of Theorem 1.7 where $\phi$ does not contain any twist part.
Theorem 6.3. Let $\phi \in \text{Mod}(S)$ be reducible and $k$ be a positive number such that $\phi^k = \prod_{j} \phi_j$ where $\phi_j = \phi^k_{|PS_j}$ is pseudo-Anosov on $PS_j$. Then for any $X,Y \in \text{Teich}(S)$, the limit geodesic ray of geodesics $\{g(X,\phi^{kn} \circ Y)\}$ $c:[0,\infty) \to \text{Teich}(S)$ in Proposition 6.1 satisfies that for any simple closed curve $\alpha \in \partial(\cup_jPS_j)$, we have

$$\lim_{t \to +\infty} \ell_{\alpha}(c(t)) = 0.$$ 

Remark 6.2. It is not hard to see the limit geodesic ray $c(\mathbb{R}^{\geq 0})$ goes to some stratum. More precisely, there exists a simple closed curve $\alpha \in \partial(\cup_jPS_j)$ such that

$$\lim_{t \to +\infty} \ell_{\alpha}(c(t)) = 0.$$

Proof of Theorem 6.3. We argue by contradiction. Assume not. Proposition 6.1 tells that there exists a positive number $\epsilon$ such that $c(\mathbb{R}^{\geq 0})$ lie in the thick part $\text{Teich}(S)_{\geq \epsilon}$. In particular, $c(\mathbb{R}^{\geq 0})$ is recurrent. Let $r:(-\infty, +\infty) \to \text{Teich}(S)$ be an axis for $\phi^k$. Since the Hausdorff distance between $c(\mathbb{R}^{\geq 0})$ and $r(\mathbb{R}^{\geq 0})$ is finite, from Proposition 5.1 we know that $r(\mathbb{R}^{\geq 0})$ is strongly asymptotic to $c(\mathbb{R}^{\geq 0})$. In particular $r(\mathbb{R}^{\geq 0}) \subset \text{Teich}(S)_{\geq \epsilon}$ which is a contradiction because $r(\mathbb{R}^{\geq 0})$ is contained in a stratum. Theorem 6.3 holds for arbitrary simple closed curve in $\partial(\cup_jPS_j)$.

Assume not. Proposition 6.1 tells that there exists a positive number $\epsilon$ such that $c(\mathbb{R}^{\geq 0})$ lie in the thick part $\text{Teich}(S)_{\geq \epsilon}$. In particular, $c(\mathbb{R}^{\geq 0})$ is recurrent. Let $r:(-\infty, +\infty) \to \text{Teich}(S)$ be an axis for $\phi^k$. Since $\ell_{\beta}$ is convex in $\text{Teich}(S)$ (see Theorem 2.1) and $c(\mathbb{R}^{\geq 0})$ is the limit geodesic ray of $\{g(X,r(n))\}_{n \geq 1}$, the length function $\ell_{\beta}$ is decreasing along $c(\mathbb{R}^{\geq 0})$. So equation (33) tells that there exists a positive number $C$ such that

$$\lim_{t \to +\infty} \ell_{\beta}(c(t)) = C > 0.$$

Since $r(\mathbb{R}) \subset T_{\beta}$, Proposition 2.11 and equation (33) tell that there exists a positive number $C_1$ such that

$$\inf_{t \to +\infty} \text{dist}(c(t), r(\mathbb{R})) = C_1 > 0.$$ 

Consider the sequence of geodesic quadrilaterals $\{\Lambda_n\}_{n \geq 1}$ where $\Lambda_n$ is the geodesic quadrilateral whose vertices are $\{r(knL_{WP}(\phi)), r(k(n+1)L_{WP}(\phi)), c(k(n+1)L_{WP}(\phi)), c(knL_{WP}(\phi))\}$. From equation (34) and Lemma 6.2 we know that $\{\Lambda_n\}_{n \geq 1}$ converges to a flat parallelogram as $n \to \infty$, which is isometric to a flat parallelogram in the two-dimensional Euclidean space $\mathbb{R}^2$. The pulled-back quadrilaterals $\{\Lambda'_n = \phi^{-kn} \circ \Lambda_n\}$ have a common edge $g(r(0), r(kL_{WP}(\phi)))$. We consider a sequence of geodesics $\{\phi^{-kn} \circ c(knL_{WP}(\phi))\} \{g(r(0), \phi^{-kn} \circ c(knL_{WP}(\phi)))\}$ from equation (34) we know that

$$\lim_{n \to +\infty} \ell(g(r(0), \phi^{-kn} \circ c(knL_{WP}(\phi)))) = C_1 < \infty.$$ 

We denote the geodesic $g(r(0), \phi^{-kn} \circ c(knL_{WP}(\phi)))$ by $g_n$. Let $\sigma$ be a simplex with $\sigma^0 = \partial(\cup_jPS_j)$. From Wolpert’s Compactness theorem (see Theorem 3.2), after passing a subsequence of $\{g_n\}$, there exists a positive number $t_1$, and a point $Z_1$ and a simplex $\sigma_1$ and a sequence of product Dehn-twists $\tau_n \in Tw(\sigma^0 - \sigma^0 \cap \sigma_1^0)$
such that $Z_1 \in T_{\sigma_1}$ and $\tau_n \circ g(r(0), g_n(t_1))$ converges in $\overline{\text{Teich}(S)}$ to the geodesic $g(r(0), Z_1)$. In particular $\tau_n \circ g_n(t_1) \to Z_1$ as $n \to +\infty$.

Since $\tau_n \in Tw[\sigma^0 - \sigma^0 \cap \sigma^0]$, $\tau_n$ fixes the geodesic segment $g(r(0), r(kL_{WP}(\phi)))$. Since the geodesic quadrilaterals $\{\Lambda_n\}_{n \geq 1}$ converge to a flat parallelogram as $n \to \infty$, so does $\{\Lambda'_n = \phi^{-kn} \circ \Lambda_n\}$ and $\tau_n \circ \Lambda'_n$. Since $\tau_n \circ \Lambda'_n$ converges to a flat quadrilateral, the geodesic triangles with vertices $\{r(0), r(kL_{WP}(\phi)), \tau_n \circ g_n(t_1)\}$, contained in $\tau_n \circ \Lambda'_n$, also converge to a geodesic flat triangle which is isometric to a flat triangle in $\mathbb{R}^2$. Since $\tau_n \circ g_n(t_1) \to Z_1$ as $n \to \infty$, the limit geodesic triangle with vertices $\{r(0), r(kL_{WP}(\phi)), Z_1\}$ is a flat geodesic triangle in $\overline{\text{Teich}(S)}$. Recall that $Z_1 \in T_{\sigma_1}$.

Claim: $\sigma^0_1 \subset \cup_j PS_j$ and $\sigma_1 \neq \sigma$.

If the Claim is correct, from Lemma 2.13 we get a contradiction because of the existence of the flat geodesic triangle $\{\Delta(r(0), r(kL_{WP}(\phi)), Z_1)\}$. Then the conclusion follows.

Proof of the Claim. First we show $\sigma_1 \neq \sigma$.

If not. That is, $\sigma_1 = \sigma$. Then we get $k = 1$ in Wolpert’s compactness theorem (see Theorem 3.2). So $\tau_n$ is trivial, $t_1 = C_1$, and

$$\lim_{n \to +\infty} \phi^{-kn} \circ c(knL_{WP}(\phi)) = Z_1 \in T_{\sigma}.$$  

On the other hand, from our assumption on equation (33) we have

$$\lim_{n \to +\infty} \ell(\phi^{-kn} \circ c(knL_{WP}(\phi))) = 0.$$  

Since $\phi$ fixes $\beta$, we have for all $n \geq 1$,

$$\ell(\phi^{-kn} \circ c(knL_{WP}(\phi))) = \ell(c(knL_{WP}(\phi))).$$  

Since the length function is smooth, after taking a limit on equation (38) as $n \to \infty$, equation (36) and inequality (37) tell that

$$\ell(\beta(Z_1)) > 0$$

which contradicts the fact $Z_1 \in T_{\sigma}$.

Secondly we show that $\sigma^0_1 \subset \cup_j PS_j$.

For any non-peripheral essential simple closed curve $\gamma \in \sigma^0_1$ with $\gamma \notin \cup_j PS_j$, then either $\gamma$ is a non-peripheral essential simple closed curve in the complement $S - \cup_j PS_j$ or $\gamma$ is a non-peripheral essential simple closed curve intersecting with $\cup_j \partial(PS_j)$. We are going to show that no curve in these two cases pinches to zero along the limit geodesic ray.

Case (a). Assume that $\gamma \in \sigma^0_1$ is a non-peripheral essential simple closed curve in the complement $S - \cup_j PS_j$ of $\cup_j PS_j$. Then it is not hard to see that there exists another non-peripheral essential simple closed curve $\gamma'$ which is also in the complement $S - \cup_j PS_j$ of $\cup_j PS_j$ and intersects with $\gamma$. Since $\phi$ fixes $\gamma'$, Part (2) of Proposition 6.1 tells that there exists a constant $\epsilon > 0$ such that for all $n \geq 1$,

$$\ell(\phi^{-kn} \circ c(knL_{WP}(\phi))) \leq \frac{1}{\epsilon}.$$
Theorem 2.1 tells that
\[
\max_{Z \in g(r(0), \phi^{-k} \circ (knL_{WP}(\phi))} \ell_{\gamma'}(Z) \leq \max\{\ell_{\gamma'}(r(0)), \frac{1}{\epsilon}\}
\]
\[
< +\infty.
\]

Since \( \gamma \) intersects with \( \gamma' \), it follows from the Collar Lemma (see [Bus10]) that there exists a positive number \( \epsilon_1 \) such that for all \( n \geq 1 \) we have
\[
\max_{Z \in g(r(0), \phi^{-k} \circ (knL_{WP}(\phi))}} \ell_{\gamma}(Z) \geq \epsilon_1 > 0.
\]

In particular, for all \( n \geq 1 \) we have
\[
\ell_{\gamma}(g_n(t_1)) \geq \epsilon_1 > 0.
\]

Recall \( \tau_n \in Tw(\sigma^0 - \sigma^0 \cap \sigma^0_1) \) and \( \gamma \subset (S - \cup_j PS_j) \). So \( \tau_n \) fixes \( \gamma \). Hence, Inequality (40) tells that for all \( n \geq 1 \) we have
\[
\ell_{\gamma}(\tau_n \circ g_n(t_1)) = \ell_{\gamma}(g_n(t_1)) \geq \epsilon_1 > 0.
\]

Since the length function is smooth and \( \tau_n \circ g_n(t_1) \to Z_1 \) as \( n \to \infty \), after taking a limit on inequality (41) as \( n \to \infty \), we have
\[
\ell_{\gamma}(Z_1) \geq \epsilon_1 > 0
\]
which contradicts the assumption \( Z_1 \in T\sigma_1 \).

Case (b). Assume that \( \gamma \in \sigma^0_1 \) is a non-peripheral essential simple closed curve intersecting with \( \cup_j \partial(PS_j) \). Let \( \alpha \in \cup_j \partial(PS_j) \) such that \( \alpha \) intersects with \( \gamma \). Since \( \phi \) fixes \( \alpha \), we have for all \( n \geq 1 \),
\[
\ell_{\alpha}(\phi^{-k} \circ c(knL_{WP}(\phi))) = \ell_{\alpha}(c(knL_{WP}(\phi))).
\]

Theorem 2.1 tells that for all \( n \geq 1 \) we have
\[
\max_{Z \in g(r(0), \phi^{-k} \circ (knL_{WP}(\phi))}} \ell_{\alpha}(Z) \leq \max\{\ell_{\alpha}(r(0)), \ell_{\alpha}(c(knL_{WP}(\phi)))\}
\]
\[
\leq \ell_{\alpha}(c(knL_{WP}(\phi))).
\]

Since \( \ell_{\alpha} \) is decreasing along the geodesic ray \( c(\mathbb{R}^{\geq 0}) \) and \( c(0) = X \), we have for all \( n \geq 1 \),
\[
\ell_{\alpha}(c(knL_{WP}(\phi))) \leq \ell_{\alpha}(X) < \infty.
\]

In particular, for all \( n \geq 1 \) we have
\[
\ell_{\alpha}(g_n(t_1)) \leq \ell_{\alpha}(X).
\]
Recall $\tau_n \in Tw(\sigma^0 - \sigma^0 \cap \sigma^0_1)$ and $\alpha \in \sigma^0$. So $\tau_n$ fixes $\alpha$. Hence, Inequality (43) tells that
\begin{align*}
\ell_{\alpha}(\tau_n \circ g_n(t_1)) &= \ell_{\tau_n \circ \alpha}(g_n(t_1)) \\
&= \ell_{\alpha}(\tau_n(g_n(t_1))) \\
&\leq \ell_{\alpha}(X) \\
&< \infty.
\end{align*}

Since $\gamma$ intersects with $\alpha$, it follows from inequality (44) and the Collar lemma (see [Bus10]) that there exists a positive number $\epsilon_2$ such that
\begin{equation}
\ell_{\gamma}(\tau_n \circ g_n(t_1)) \geq \epsilon_2 > 0.
\end{equation}

Therefore, any simple closed curve, whose length pinches to zero along any limit of geodesics $\{g(r(0), \phi^{-kn} \circ c(knL_{WP}(|\phi|)))\}$ in the sense of Theorem 3.2, is contained in $\bigcup_j PS_j$. In particular, we have
\begin{equation}
\sigma^0_1 \subset \bigcup_j PS_j.
\end{equation}

Now we are ready to prove Theorem 1.7.

Proof of Theorem 1.7. Proof of Part (1). Since $L_{WP}(\psi) > 0$, Theorem 2.4 tells that there exists a geodesic line $r : (-\infty, +\infty) \to \overline{\text{Teich}(S)}$ such that $\phi^n \circ r(0) = r(nL_{WP}(\psi))$ for all $n \in \mathbb{Z}$. Since $\text{Mod}(S)$ acts on $\overline{\text{Teich}(S)}$ as isometries, we have for all $n \in \mathbb{Z}$,
\begin{equation*}
\text{dist}(\phi^n \circ Y, \phi^n \circ r(0)) = \text{dist}(Y, r(0)) < +\infty.
\end{equation*}

Hence, the geodesics $\{g(X, \phi^n \circ Y)\}$ converge to a geodesic ray $c : [0, +\infty) \to \overline{\text{Teich}(S)}$ emanating from $X$ which has finite Hausdorff distance to $r(\mathbb{R}_{\geq 0})$ (see the proof of Proposition 8.2 in [BH99]). From Theorem 2.3 we know that $c(\mathbb{R}_{\geq 0}) \subset \text{Teich}(S)$.

Proof of Part (2). Since the geodesics $\{g(X, \phi^n \circ Y)\}$ is convergent, the subsequence $\{g(X, \phi^{kn} \circ Y)\}$ also converges to $c : [0, +\infty) \to \overline{\text{Teich}(S)}$. Set $\phi' = \prod_j \phi_j$. Then we have for all $n \geq 1$,
\begin{equation}
\text{dist}(\phi^{kn} \circ Y, \phi^n \circ Y) = \text{dist}(\prod_{\alpha \in \sigma^0_1} \tau_n \circ Y, Y).
\end{equation}

Equation (47) and Theorem 1.4 tell that there exists a positive number $C$ such that for all $n \geq 1$,
\begin{equation*}
\text{dist}(\phi^{kn} \circ Y, \phi^n \circ Y) \leq C.
\end{equation*}

So the limit geodesic ray of the geodesics $\{g(X, \phi^n \circ Y)\}$ is the same as the limit ray of the geodesics $\{g(X, \phi^n \circ Y)\}$ as $n \to \infty$ (see the proof of Proposition 8.2 in
[BH99]), which is also $c : [0, +\infty) \to \text{Teich}(S)$. From Theorem 6.3, we have for any simple closed curve $\alpha \in \partial(\cup_j \partial_j)$,

$$\lim_{t \to +\infty} \ell_\alpha(c(t)) = 0.$$ 

Proof of Part (3). Since $c : [0, +\infty) \to \text{Teich}(S)$ is also the limit ray of the geodesics $\{g(X, \phi^n \circ Y)\}$, from Proposition 6.1 we know that there exists a positive number $\epsilon_0$ such that

$$\inf_{t \geq 0} \ell_\beta(c(t)) \geq \epsilon_0$$

for any non-peripheral essential simple closed curve $\beta \notin \partial(\cup_j \partial_j)$.

□

References


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