2. THE WEYL CRITERION

PROOF. We recall the following theorem of Frobenius (see Knopp [1, pp. 507-508] and G. M. Petersen [2, pp. 48-49]). Let \((c_n), n = 1, 2, \ldots\), be a sequence of complex numbers with the property that
\[
\lim_{N \to \infty} \frac{1}{N}(c_1 + \cdots + c_N)
\]
exists and equals \(c\). Set \(\Phi(r) = \sum_{n=1}^{\infty} c_n r^n\); then we have \(\lim_{r \to 1^-} (1 - r)\Phi(r) = c\).

Now we turn to the proof of our theorem. We have, according to the definition of \(G(z)\),
\[
G(re^{2\pi im\alpha}) = \sum_{n=1}^{\infty} f(n\alpha)e^{2\pi inm\alpha}r^n
\]
for \(-1 < r < 1\) and \(m \in \mathbb{Z}\).

Since \((n\alpha)\) is u.d. mod 1, one obtains from Corollary 1.1 the relation
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha)e^{2\pi inm\alpha} = \int_{0}^{1} f(x)e^{2\pi imx} dx = d_m, \text{ say.}
\]

Using the theorem of Frobenius, we get
\[
\lim_{r \to 1^-} (1 - r)G(re^{2\pi im\alpha}) = d_m.
\]
For \(m \geq m_0\) we have \(d_m \neq 0\). So (2.11) implies that if \(z \to e^{2\pi im\alpha}\) along the radius, the function value \(G(z)\) tends to \(\infty\), and therefore, \(G\) has singularities on an everywhere dense subset of \(\{z \in \mathbb{C} : |z| = 1\}\).

Fejér's Theorem

As another consequence of the Weyl criterion, we obtain a theorem that will provide many more examples of sequences that are u.d. mod 1.

THEOREM 2.5. Let \((f(n)), n = 1, 2, \ldots\), be a sequence of real numbers such that \(\Delta f(n) = f(n + 1) - f(n)\) is monotone as \(n\) increases. Let, furthermore,
\[
\lim_{n \to \infty} \Delta f(n) = 0 \quad \text{and} \quad \lim_{n \to \infty} n |\Delta f(n)| = \infty.
\]
Then the sequence \((f(n))\) is u.d. mod 1.

PROOF. For every pair of real numbers \(u\) and \(v\) we have
\[
|e^{2\pi iu} - e^{2\pi iv} - 2\pi i(u - v)e^{2\pi iv}| = |e^{2\pi i(u - v)} - 1 - 2\pi i(u - v)|
\]
\[
= 4\pi^2 \left| \int_{0}^{u-v} (u - v - w)e^{2\pi i w} dw \right|
\]
\[
\leq 4\pi^2 \left| \int_{0}^{u-v} (u - v - w) \right| dw
\]
\[
= 2\pi^2(u - v)^2.
\]
Now set \( u = hf(n + 1) \) and \( v = hf(n) \), where \( h \) is a nonzero integer. Then, according to (2.13),

\[
\left| \frac{e^{2\pi ihf(n+1)}}{\Delta f(n)} - \frac{e^{2\pi ihf(n)}}{\Delta f(n)} - 2\pi ihe^{2\pi ihf(n)} \right| \leq 2\pi^2h^2|\Delta f(n)| \quad \text{for } n \geq 1;
\]

hence,

\[
\left| \frac{e^{2\pi ihf(n+1)}}{\Delta f(n+1)} - \frac{e^{2\pi ihf(n)}}{\Delta f(n)} - 2\pi ihe^{2\pi ihf(n)} \right| \leq \left| \frac{1}{\Delta f(n)} - \frac{1}{\Delta f(n+1)} \right| + 2\pi^2h^2|\Delta f(n)| \quad \text{for } n \geq 1. \tag{2.14}
\]

Then,

\[
\left| 2\pi ihe^{2\pi ihf(n)} \right| = \left| \sum_{n=1}^{N-1} \left( \frac{e^{2\pi ihf(n+1)}}{\Delta f(n+1)} - \frac{e^{2\pi ihf(n)}}{\Delta f(n)} + \frac{e^{2\pi ihf(N)}}{\Delta f(N)} - \frac{e^{2\pi ihf(1)}}{\Delta f(1)} \right) \right|
\leq \sum_{n=1}^{N-1} \left| \frac{1}{\Delta f(n)} - \frac{1}{\Delta f(n+1)} \right| + 2\pi^2h^2\sum_{n=1}^{N-1}|\Delta f(n)| + \frac{1}{|\Delta f(N)|} + \frac{1}{|\Delta f(1)|},
\]

where we used (2.14) in the last step. Because of the monotonicity of \( \Delta f(n) \), we get

\[
\left| \frac{1}{N} \sum_{n=1}^{N-1} e^{2\pi ihf(n)} \right| \leq \frac{1}{\pi |h|} \left( \frac{1}{N|\Delta f(1)|} + \frac{1}{N|\Delta f(N)|} \right) + \frac{\pi |h|}{N} \sum_{n=1}^{N-1}|\Delta f(n)|,
\]

and therefore, in view of (2.12),

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N-1} e^{2\pi ihf(n)} = 0.
\]

**COROLLARY 2.1:** Fejér's Theorem. Let \( f(x) \) be a function defined for \( x \geq 1 \) that is differentiable for \( x \geq x_0 \). If \( f'(x) \) tends monotonically to 0 as \( x \to \infty \) and if \( \lim_{x \to \infty} x|f'(x)| = \infty \), then the sequence \( (f(n)), n = 1, 2, \ldots, \) is u.d. mod 1.

**PROOF.** The mean value theorem shows that \( \Delta f(n) \) satisfies the conditions of Theorem 2.5, at least for sufficiently large \( n \). The finitely many exceptional terms do not influence the u.d. mod 1 of the sequence.

**EXAMPLE 2.7.** Fejér's theorem immediately implies the u.d. mod 1 of the following types of sequences: (i) \((\alpha n^\gamma \log^r n), n = 2, 3, \ldots, \) with \( \alpha \neq 0, \)
0 < \sigma < 1, and arbitrary \tau; (ii) \((\alpha \log^\tau n), n = 1, 2, \ldots\), with \(\alpha \neq 0\) and \(\tau > 1\); (iii) \((\alpha n \log^\tau n), n = 2, 3, \ldots\), with \(\alpha \neq 0\) and \(\tau < 0\).

The following simple result shows that the second condition in (2.12) cannot be relaxed too much.

**THEOREM 2.6.** If a sequence \((f(n)), n = 1, 2, \ldots\), is u.d. mod 1, then necessarily \(\lim_{n \to \infty} n |\Delta f(n)| = \infty\).

**PROOF.** Suppose that \((f(n))\) is u.d. mod 1 and that \(\lim_{n \to \infty} n |\Delta f(n)| < \infty\). For any two real numbers \(u\) and \(v\), we have

\[
|e^{2\pi i u} - e^{2\pi i v}| \leq 2\pi |u - v|,
\]

and so,

\[
|e^{2\pi i f(n+1)} - e^{2\pi i f(n)}| \leq 2\pi |\Delta f(n)| = O\left(\frac{1}{n}\right).
\]

On the other hand,

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i f(n)} = 0.
\]

By a well-known Tauberian theorem (Hardy [2, p. 121], G. M. Petersen [2, p. 51]) it follows that \(\lim_{n \to \infty} e^{2\pi i f(n)} = 0\), an obvious absurdity.

**An Estimate for Exponential Sums**

Although Corollary 2.1 is a very powerful result, there are various interesting sequences to which it does not apply. For instance, the question whether \((n \log n), n = 1, 2, \ldots\), is u.d. mod 1 cannot be settled by appealing to Fejér's theorem. In such cases, the following estimate may prove to be useful. We first need some technical lemmas. The values of the absolute constants will not be important in these estimates.

**LEMMA 2.1.** Suppose the real-valued function \(f\) has a monotone derivative \(f'\) on \([a, b]\) with \(f'(x) \geq \lambda > 0\) or \(f'(x) \leq -\lambda < 0\) for \(x \in [a, b]\). Then, if \(J = \int_{a}^{b} e^{2\pi i f(x)} \, dx\), we have \(|J| < 1/\lambda\).

**PROOF.** We have

\[
J = \frac{1}{2\pi i} \int_{a}^{b} \frac{de^{2\pi i f(x)}}{f'(x)},
\]

and therefore, an application of the second mean value theorem yields, with some \(x_0 \in [a, b]\),

\[
|J| = \left| \frac{1}{2\pi i} \left( \frac{1}{f'(a)} \int_{a}^{x_0} e^{2\pi i f(x)} \, dx + \frac{1}{f'(b)} \int_{x_0}^{b} e^{2\pi i f(x)} \, dx \right) \right|
\]

\[
\leq \frac{1}{2\pi} \left( \frac{2}{|f'(a)|} + \frac{2}{|f'(b)|} \right) \leq \frac{2}{\pi \lambda} < \frac{1}{\lambda}. \quad \blacksquare
\]
LEMMA 2.2. Let $f$ be twice differentiable on $[a, b]$ with $f''(x) \geq \rho > 0$ or $f''(x) \leq -\rho < 0$ for $x \in [a, b]$. Then the integral $J$ from Lemma 2.1 satisfies $|J| < 4/\sqrt{\rho}$.

PROOF. We may suppose that $f''(x) \geq \rho$ for $x \in [a, b]$; otherwise, we replace $f$ by $-f$. We note that $f'$ is increasing. Suppose for the moment that $f'$ is of constant sign in $[a, b]$, say $f' \geq 0$. If $a < c < b$, then $f'(x) \geq (c - a)\rho$ for $c \leq x \leq b$ by the mean value theorem. Therefore, by Lemma 2.1,

$$|J| \leq \left| \int_a^c e^{2\pi i f(x)} \, dx \right| + \left| \int_c^b e^{2\pi i f(x)} \, dx \right| < (c - a) + \frac{1}{(c - a)\rho},$$

and choosing $c$ so as to make the last sum a minimum, we obtain $|J| < 2/\sqrt{\rho}$. In the general case, $[a, b]$ is the union of two intervals in each of which $f'$ is of constant sign, and the desired inequality follows by adding the inequalities for these two intervals. ■

LEMMA 2.3. Let $f'$ be monotone on $[a, b]$ with $|f'(x)| \leq \frac{1}{2}$ for $x \in [a, b]$. Then, if $J_1 = \int_a^b \left( \frac{x}{2} - \frac{1}{2} \right) \, dx$, we have

$$|J_1| \leq 2.$$  \hspace{1cm} (2.16)

PROOF. We start from the Fourier-series expansion

$$\{x\} - \frac{1}{2} = -\sum_{h=1}^{\infty} (\sin 2\pi hx)/\pi h,$$

valid for all $x \notin \mathbb{Z}$. For $m \geq 1$, let $\chi_m(x) = -\sum_{h=1}^{m} (\sin 2\pi hx)/\pi h$, $x \in \mathbb{R}$, be the $m$th partial sum. The functions $\chi_m, m = 1, 2, \ldots$, are uniformly bounded as is seen easily after summation by parts. Therefore,

$$J_1 = \lim_{m \to \infty} \int_a^b \chi_m(x) \, dx.$$  \hspace{1cm} (2.17)

Now for $m \geq 1$ we have

$$\int_a^b \chi_m(x) \, dx = \sum_{h=1}^{m} \frac{1}{h} \int_a^b (-2i \sin 2\pi hx) e^{2\pi if(x)} f'(x) \, dx = \sum_{h=1}^{m} \frac{1}{h} \int_a^b (e^{-2\pi ihx} - e^{2\pi ihx}) e^{2\pi if(x)} f'(x) \, dx = \frac{1}{2\pi i} \sum_{h=1}^{m} \frac{1}{h} \left( \int_a^b f'(x)(x - \frac{h}{2}) e^{2\pi i f(x) - h} - \int_a^b f'(x)(x + \frac{h}{2}) e^{2\pi i f(x) + h} \right).$$
2. THE WEYL CRITERION

Since the functions \( f'(f' \pm h) \) are monotone and \(|f'| \leq \frac{1}{2} \), an application of the second mean value theorem shows that

\[
\left| \int_a^b \frac{f'(x)}{f'(x) \pm h} \, d e^{2\pi i f(x) \pm hx} \right| \leq \frac{2}{h - \frac{1}{2}},
\]

and so

\[
\left| \int_a^b \chi_m(x) \, d e^{2\pi i f(x)} \right| \leq \frac{2}{\pi} \sum_{n=1}^{m} \frac{1}{h(h - \frac{1}{2})} < 2. \tag{2.18}
\]

Equations (2.17) and (2.18) imply (2.16).

**THEOREM 2.7.** Let \( a \) and \( b \) be integers with \( a < b \), and let \( f \) be twice differentiable on \([a, b]\) with \( f''(x) \geq \rho > 0 \) or \( f''(x) \leq -\rho < 0 \) for \( x \in [a, b] \). Then,

\[
\left| \sum_{n=a}^{b} e^{2\pi i f(n)} \right| \leq (|f''(b) - f''(a)| + 2) \left( \frac{4}{\sqrt{\rho}} + 3 \right). \tag{2.19}
\]

**PROOF.** We write

\[
\sum_{n=a}^{b} e^{2\pi i f(n)} = \sum_{p=-\infty}^{\infty} S_p, \tag{2.20}
\]

with

\[
S_p = \sum_{a \leq n \leq b \atop p-1/2 \leq f''(n) < p+1/2} e^{2\pi i f(n)}. \tag{2.21}
\]

The sum over \( p \) in (2.20) is in reality just a finite sum. Let \( p \) be an integer for which the sum in (2.21) is nonvoid. Since \( f' \) is monotone, this sum is over consecutive values of \( n \), say from \( n = a_p \) to \( n = b_p \). With \( F_p(x) = f(x) - px \), we get

\[
S_p = \sum_{n=a_p}^{b_p} e^{2\pi i f(n)} = \sum_{n=a_p}^{b_p} e^{2\pi i F_p(n)}
\]

\[
= \int_{a_p}^{b_p} e^{2\pi i F_p(x)} \, dx + \frac{1}{2}(e^{2\pi i F_p(a_p)} + e^{2\pi i F_p(b_p)})
\]

\[
+ \int_{a_p}^{b_p} \{x\} \, d e^{2\pi i F_p(x)} \tag{2.22}
\]

by the Euler summation formula; compare with (2.3). Now the first integral in (2.22) is in absolute value less than \( 4/\sqrt{\rho} \) by Lemma 2.2. The second integral in (2.22) is in absolute value at most 2 because of \(|F'_p(x)| \leq \frac{1}{2} \) for \( x \in [a_p, b_p] \) and Lemma 2.3. Therefore, \(|S_p| < (4/\sqrt{\rho}) + 3 \). Since there are at most \(|f''(b) - f''(a)| + 2 \) values of \( p \) for which \( S_p \) is a nonvoid sum, we arrive at (2.19).
EXAMPLE 2.8. From Theorem 2.7 we infer that
\[ \left| \frac{1}{N} \sum_{n=1}^{N} e^{2\pi i n h \log n} \right| \leq \frac{1}{N} (|h| \log N + 2) \left(4\sqrt{\frac{N}{|h|}} + 3\right) \]

for all nonzero integers \( h \), and so, the sequence \( (n \log n) \), \( n = 1, 2, \ldots \), is u.d. mod 1 by the Weyl criterion. More generally, the method yields that \((\alpha n \log^\tau n) \), \( n = 1, 2, \ldots \), \( \alpha \neq 0, \tau > 0 \), is u.d. mod 1. In the same way, the sequence \((n \log \log n) \), \( n = 2, 3, \ldots \), can be shown to be u.d. mod 1. Compare also with Exercises 2.23–2.26. ■

Uniform Distribution of Double Sequences

DEFINITION 2.1. A double sequence \((s_{jk})\), \( j = 1, 2, \ldots \), \( k = 1, 2, \ldots \), of real numbers is said to be u.d. mod 1 if for any \( a \) and \( b \) such that \( 0 < a < b < 1 \),
\[ \lim_{M,N \to \infty} \frac{A([a, b); M, N)}{MN} = b - a, \]  

(2.23)

where \( A([a, b); M, N) \) is the number of \( s_{jk} \), \( 1 \leq j \leq M, 1 \leq k \leq N \), for which \( a < s_{jk} < b \).

THEOREM 2.8. The double sequence \((s_{jk})\) is u.d. mod 1 if and only if for every Riemann-integrable function \( f \) on \( I \) we have
\[ \lim_{M,N \to \infty} \frac{1}{MN} \sum_{j=1}^{M} \sum_{k=1}^{N} f(s_{jk}) = \int_{0}^{1} f(x) \, dx. \]

THEOREM 2.9. The double sequence \((s_{jk})\) is u.d. mod 1 if and only if
\[ \lim_{M,N \to \infty} \frac{1}{MN} \sum_{j=1}^{M} \sum_{k=1}^{N} e^{2\pi i hs_{jk}} = 0 \quad \text{for all integers } h \neq 0. \]

The proofs of these theorems can be given along the same lines as those of Corollary 1.1 and Theorem 2.1, respectively.

EXAMPLE 2.9. Let \( \theta \) be irrational and \( \alpha \) an arbitrary real number. Then \((j\theta + k\alpha) \), \( j = 1, 2, \ldots \), \( k = 1, 2, \ldots \), is u.d. mod 1, as follows easily from Theorem 2.9. ■

Any u.d. double sequence \((s_{jk})\) can be arranged into a u.d. single sequence \((s_{n})\), say. One just has to choose \((s_{n})\) in such a way that its first \( M^2 \) terms consist exactly of all numbers \( s_{jk} \) with \( 1 \leq j, k \leq M \). For instance, the arrangement \( s_{11}, s_{21}, s_{12}, s_{31}, s_{22}, s_{32}, s_{33}, s_{41}, \ldots, s_{14}, \ldots \) satisfies this property. To prove our assertion, we note that for a given positive integer \( m \) there exists a unique integer \( M \) with \((M - 1)^{2} \leq m < M^{2} \), which implies