ACTIONS OF SOME POINTED HOPF ALGEBRAS ON PATH ALGEBRAS OF QUIVERS

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Goal:

To understand examples of Hopf Actions on Algebras

We say that a Hopf algebra H acts on an algebra A if

A is an H-*module algebra*:

A is an H-module, and the multiplication and unit maps of A are H-morphisms.

We also need a notion of faithfulness:

H acts on A *inner faithfully*

if there is not an induced action of H/I on A for any nonzero Hopf ideal I of H.

In other words, the Hopf action does not factor through a smaller Hopf quotient.

Two types of results

Fix a field $\Bbbk$.
Let $\mathcal{H}$ be a class of Hopf algebras over $\Bbbk$.
Let $\mathcal{A}$ be a class of algebras over $\Bbbk$.

[No Quantum Symmetry]

If $H \in \mathcal{H}$ acts inner faithfully on any $A \in \mathcal{A}$,
then H must be cocommutative.

(e.g. the Hopf action factors through the action of a cocommutative Hopf algebra)

[Honest Quantum Symmetry]

Classify all pairs (H, A) so that $H \in \mathcal{H}$ acts inner faithfully on $A \in \mathcal{A}$.
Here, at least one H is non-cocommutative.

This problem is more tractable when either:
the size of the class of Hopf algebras $\mathcal{H}$ is limited, or
the size of the class of algebras $\mathcal{A}$ is limited.

Our Setting

[Honest Quantum Symmetry] with $\mathcal{H}$ limited, $\mathcal{A}$ vast

$\Bbbk$ = containing a primitive n-th root of unity ζ (char $\Bbbk$ is coprime to n)

H = Taft algebra $T(n)$

generated by grouplike element g and $(1, g)$-skew primitive element x,
subject to relations: $g^n = 1, x^n = 0$, and $xg = \zeta gx$

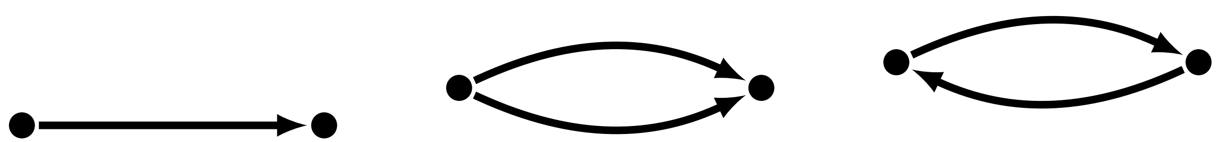
($H = u_q(\mathfrak{sl}_2)$ for q a primitive $2n$-th root of unity, or $D(T(n))$, later)

$\mathcal{A}$ = path algebras $\Bbbk Q$, of a quiver Q

Q is a directed graph consisting of a set of vertices Q_0, a set of arrows Q_1,
and start/target maps $s/t : Q_1 \to Q_0$.
Basis elements of $\Bbbk Q$ are paths in Q. Multiplication of basis elements
is the composition of paths where defined, or 0 otherwise.

Standing Hypotheses

- The quiver Q is finite ($|Q_0|, |Q_1| < \infty$)
- Q is loopless
- Q is Schurian ($\forall\ i, j \in Q_0$, $\exists$ at most one $a \in Q_1$ with $s(a) = i$ and $t(a) = j$)



Schurian　　Not Schurian　　Schurian

- The action of $T(n)$ preserves the path length filtration on $\Bbbk Q$

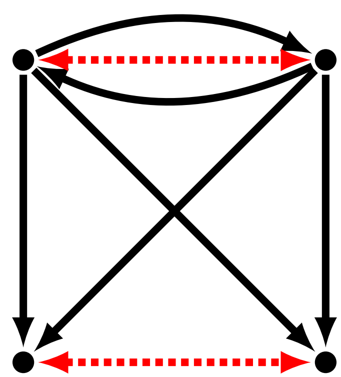
  (e.g. for $x \in T(n)$ and $a \in Q_1$, we allow $x \cdot a \in \Bbbk Q_0$)

Theorem 1 [$T(n)$-actions on $\Bbbk Q$]

Given any quiver Q that admits a faithful action of $\mathbb{Z}_n$
(by quiver automorphism),

we have a classification of (e.g. precise formulae for)
inner faithful actions of $T(n)$ on $\Bbbk Q$
that extend the given $\mathbb{Z}_n$-action on Q.

Example: We classify Sweedler ($T(2)$)-actions on the path algebra of Q below.



Here, the action of $\mathbb{Z}_2$ is given by 

Steps of Proof of Theorem 1

I. Decompose Q into a certain union of subquivers $\{Q^\ell\}$ so that $Q^\ell \cap Q^{\ell'} \subset Q_0$ for $\ell \neq \ell'$.

II. Have explicit formulae for $T(n)$-action on $\Bbbk Q^\ell$

III. We obtain $T(n)$-action on $\Bbbk Q$ from the set of $T(n)$-actions on $\Bbbk Q^\ell$, by making identifications of the vertices in intersections $Q^\ell \cap Q^{\ell'}$ with $\ell \neq \ell'$.

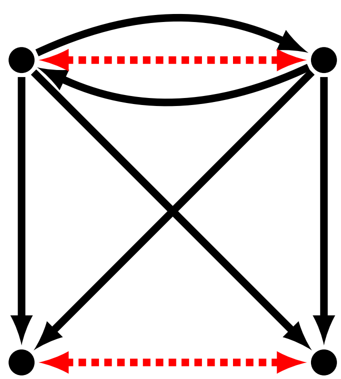
Further, the *inner faithful* actions of $T(n)$ on $\Bbbk Q$ are those for which x does not act by zero.

Steps of Proof of Theorem 1

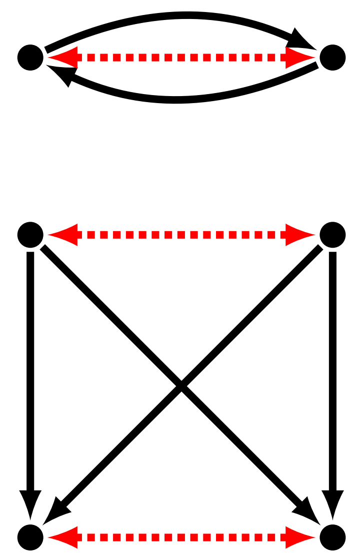
I. Decompose Q into a certain union of subquivers $\{Q^\ell\}$ w/ $Q^\ell \cap Q^{\ell'} \subset Q_0$ for $\ell \neq \ell'$.

- We can take each Q^ℓ to be a $\mathbb{Z}_n$-stable subquiver of a complete digraph or complete bipartite graph, that is *maximal* with respect to partial ordering given by inclusion.
- Call such Q^ℓ a $\mathbb{Z}_n$-component of Q.
- Such a decomposition of Q is unique up to relabelling.

Example continued:



Q (that admits faithful $\mathbb{Z}_2$-action)



$\mathbb{Z}_2$-components of Q

Steps of Proof of Theorem 1

II. Have explicit formulae for $T(n)$-action on $\Bbbk Q^\ell$

- Take e_i to be the trivial path at $i \in Q_0$, and take $\mathbb{Z}_n = \langle g \rangle$.

- Take $\mathcal{O} = \{1, \dots, m\}$ to be a $\mathbb{Z}_n$-orbit of vertices, for $m|n$.

- Relabel vertices so that $g \cdot e_i = e_{i+1}$, with indices taken modulo m.

- We get that $x \cdot e_i = \gamma \zeta^i (e_i - \zeta e_{i+1})$ for $\gamma \in \Bbbk$

- The $\mathbb{Z}_n$-action on arrows is given by quiver automorphism, up to scalar multiple.

- For $a \in (Q^\ell)_1$, we get that $x \cdot a = \alpha a + \beta(g \cdot a) + \lambda \sigma(a)$,
  where $\alpha, \beta, \lambda \in \Bbbk$ depends on the configuration of a and $g \cdot a$, and
  $\sigma(a)$ is a path with start $= s(a)$ and target $= t(g \cdot a)$ (if it exists, or 0 otherwise).

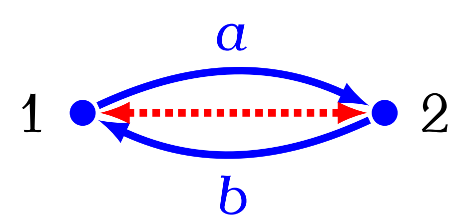
We illustrate this with the running Example. Brace yourself!

Steps of Proof of Theorem 1

II. Have explicit formulae for $T(n)$-action on $\Bbbk Q^\ell$

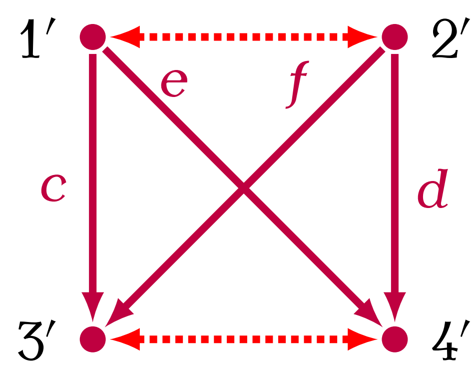
$\sigma(a) = e_1 \quad \sigma(b) = e_2$

$$\boxed{\begin{array}{l} g \cdot e_1 = e_2 \quad g \cdot e_2 = e_1 \\ g \cdot a = \mu b \quad g \cdot b = \mu^{-1} a \\ x \cdot e_1 = -\gamma(e_1 + e_2) \quad x \cdot e_2 = \gamma(e_1 + e_2) \\ x \cdot a = \gamma a - \gamma\mu b + \lambda e_1 \\ x \cdot b = -\gamma b + \gamma\mu^{-1} a - \lambda\mu^{-1} e_2 \end{array}}$$



$\sigma(c) = e \quad \sigma(d) = f \quad \sigma(e) = c \quad \sigma(f) = d$

$$\boxed{\begin{array}{l} g \cdot e_{1'} = e_{2'} \quad g \cdot e_{2'} = e_{1'} \\ g \cdot e_{3'} = e_{4'} \quad g \cdot e_{4'} = e_{3'} \\ g \cdot c = \mu' d \quad g \cdot d = \mu'^{-1} c \\ g \cdot e = \mu'' f \quad g \cdot f = \mu''^{-1} e \\ x \cdot e_{1'} = -\gamma'(e_{1'} + e_{2'}) \quad x \cdot e_{2'} = \gamma'(e_{1'} + e_{2'}) \\ x \cdot e_{3'} = -\gamma''(e_{3'} + e_{4'}) \quad x \cdot e_{4'} = \gamma''(e_{3'} + e_{4'}) \\ x \cdot c = -\gamma'' c - \gamma'\mu' d + \lambda' e \\ x \cdot d = \gamma'' d + \gamma'\mu'^{-1} c - \lambda'\mu'^{-1}\mu'' f \\ x \cdot e = \gamma'' e - \gamma'\mu'' f + \lambda'' c \\ x \cdot f = -\gamma'' f + \gamma'\mu''^{-1} e - \lambda'\mu'\mu''^{-1} d \end{array}}$$



for $\gamma, \gamma', \gamma'', \lambda, \lambda', \lambda'' \in \Bbbk, \quad \mu, \mu', \mu'' \in \Bbbk^\times, \quad$ with $(\gamma')^2 = (\gamma'')^2 + \lambda'\lambda''$

(You’ll remember all of those details, of course)

Steps of Proof of Theorem 1

III. We obtain the $T(n)$-action on $\Bbbk Q$ from the set of $T(n)$-actions on $\Bbbk Q^{\ell}$, by making identifications of the vertices in intersections $Q^{\ell} \cap Q^{\ell'}$.

In the running Example, identity the pairs of vertices

$$\boxed{1 \ \& \ 1'} \qquad \text{and} \qquad \boxed{2 \ \& \ 2'}$$

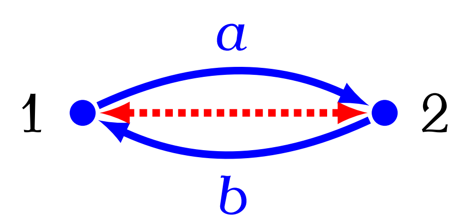
of the $\mathbb{Z}_2$-components of Q, to yield the quiver Q.

Steps of Proof of Theorem 1

II. Have explicit formulae for $T(n)$-action on $\mathbb{k}Q^{\ell}$

$\sigma(a) = e_1 \qquad \sigma(b) = e_2$

$$\boxed{\begin{array}{l} g\cdot e_1 = e_2 \qquad g\cdot e_2 = e_1 \\ g\cdot a = \mu b \qquad g\cdot b = \mu^{-1}a \\ x\cdot e_1 = -\gamma(e_1+e_2) \qquad x\cdot e_2 = \gamma(e_1+e_2) \\ x\cdot a = \gamma a - \gamma\mu b + \lambda e_1 \\ x\cdot b = -\gamma b + \gamma\mu^{-1}a - \lambda\mu^{-1}e_2 \end{array}}$$



$\sigma(c) = e \qquad \sigma(d) = f \qquad \sigma(e) = c \qquad \sigma(f) = d$

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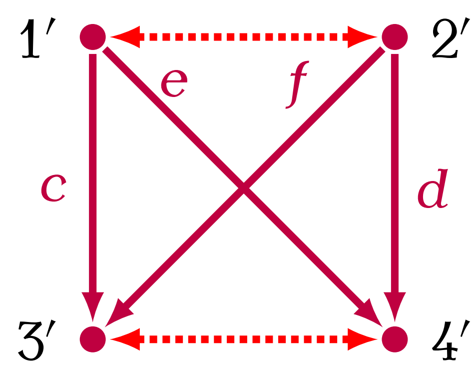

for $\gamma,\gamma',\gamma'',\lambda,\lambda',\lambda'' \in \mathbb{k}, \quad \mu,\mu',\mu'' \in \mathbb{k}^{\times}, \quad$ with $(\gamma')^2 = (\gamma'')^2 + \lambda'\lambda''$

Steps of Proof of Theorem 1

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$$\boxed{1 \;\&\; 1'} \qquad \text{and} \qquad \boxed{2 \;\&\; 2'}$$

of the $\mathbb{Z}_2$-components of Q, to yield the quiver Q.

As a result, we must impose the following restriction on the scalar parameters of the two actions above:

$$\boxed{\gamma = \gamma'}$$

Further, the *inner faithful* actions of $T(n)$ on $\Bbbk Q$ are those for which x does not act by zero.

Actions of $u_q(\mathfrak{sl}_2)$ and $D(T(n))$

We can extend the Taft actions on $\Bbbk Q$ in Theorem 1 to actions of the following Hopf algebras:

Let q be a $2n$-th root of unity. The *Frobenius-Lustzig kernel $u_q(\mathfrak{sl}_2)$* is generated by grouplike K, $(1, K)$-skew-primitive E, and $(K^{-1}, 1)$-skew-primitive F, with relations

$$KE = q^2 EK, \quad KF = q^{-2} FK, \quad K^n = 1, \quad E^n = F^n = 0, \quad EF - FE = \frac{K - K^{-1}}{q - q^{-1}}.$$

By work of H.-X Chen (1999), the *Drinfeld double $D(T(n))$ of the n-th Taft algebra* is generated by g, x, G, X, subject to relations:

$$xg = \zeta gx, \quad GX = \zeta XG, \quad gX = \zeta Xg, \quad xG = \zeta Gx, \quad gG = Gg,$$
$$g^n = G^n = 1, \quad x^n = X^n = 0, \quad xX - \zeta Xx = \zeta(gG - 1).$$

Here, g and G grouplike , x is $(1, g)$-skew primitive, and X is $(1, G)$-skew primitive.

Theorem 2 [Extended actions of $u_q(\mathfrak{sl}_2)$, $D(T(n))$ on $\Bbbk Q$]

Since $u_q(\mathfrak{sl}_2)$ and $D(T(n))$ are both generated by Hopf subalgebras that are isomorphic to Taft algebras,

namely, take $\langle K, E\rangle$, $\langle K, F\rangle$ for $u_q(\mathfrak{sl}_2)$,
and
$\langle g, x\rangle$, $\langle G, X\rangle$ for $D(T(n))$

we have the following result.

> Fix an action of $\mathbb{Z}_n$ on a quiver Q.
> Additional restraints on parameters are determined so that the Taft actions on $\Bbbk Q$ produced in Theorem 1 extend to an action of $u_q(\mathfrak{sl}_2)$ and to an action of $D(T(n))$.

On the category of Yetter Drinfel'd modules over $T(n)$

As a consequence of Theorem 2,
we obtain that $\Bbbk Q$, in the case where Q admits $\mathbb{Z}_n$-symmetry,
is an algebra in the category of Yetter-Drinfeld modules over $T(n)$.

Motivated by the Radford-Majid biproduct construction, we ask:

> Let Q be a quiver that admits $\mathbb{Z}_n$-symmetry.
> When does $\Bbbk Q$ admit the structure of a bialgebra/ Hopf algebra in the category of Yetter-Drinfeld modules over $T(n)$?