May 2018

Representations of algebras module-finite / center w/ use of Poisson geometry

Oberwolfach

applied to PI 3-dim'l, 4-dim'l Sklyanin algebras

1704.04975 1802.06487

joint with Xingting Wang & Milen Yakimov

$\Bbbk = \mathbb{C}$ for this talk

Dichotomy in the representation theory of algebras A

$\boxed{\text{A module } \infty / \text{center}}$

- loads of irreps, most are ∞-dim'l
- sometimes folks only care about fin-dim'l ones
- often one studies (categories of (f.d.)) representations; writing down explicit reps isn't feasible

$\boxed{\text{A module finite / center}}$ ($\Rightarrow$ A PI)

- irreps are all finite dim'l,
  - dim(irrep) $\leq$ PIdeg A
  - dim(irrep) = PIdeg A most of the time
- study of explicit irreps is more feasible

A of interest is usually:
a deformation of a com. alg (e.g. $\mathcal{O}(G)$)
or a cocom coalgebra (e.g. $U(\mathfrak{g})$)
& such classical Hopf algebras are viewed as the semi classical of a family of (all) such A.

Here, A can still be viewed typically as a deformation of a com algebra, but can't really consider limits.

Instead the classical com algebra that guides its representation-theoretic behavior is $\underline{\underline{Z(A)}}$

Using the affine geometry of maxSpec(Z(A)) to study irreps of A ...

Theorem [Artin, Brown, Goodearl, DeConcini, Procesi, Small ....]

① Have well-defined surjective map

$$\varphi : \mathrm{Irrep}(A) \longrightarrow \mathrm{maxSpec}(Z(A)) =: Y$$

eq. classes of irreps of A

$$\psi : A \to \mathrm{Mat}_d(\mathbb{C}) \longmapsto \ker\psi \cap Z(A) =: m_\psi \quad \text{central character}$$

$d \leq \mathrm{PIdeg} A$

②

$$\begin{array}{ccc}
\mathrm{Irrep}(A) & \xrightarrow{\ \varphi\ } & Y \\
\shortparallel & & \shortparallel \\
\mathrm{Irrep}_{\mathrm{PIdeg}(A)}(A) & \xrightarrow{\ 1-1\ } & \mathcal{A}_A \\
\sqcup & & \sqcup \\
\mathrm{Irrep}_{<\mathrm{PIdeg}}(A) & \xrightarrow{\ \text{finite to } 1\ } & \mathcal{R}_A
\end{array}$$

$\mathcal{A}_A$ Azumaya locus of A

$\mathcal{R}_A$ Ramification locus of A

$\mathcal{A}_A$ open dense subset of Y & $\mathcal{A}_A \subseteq Y^{\text{smooth}}$

It's convenient to have that $\boxed{\mathcal{A}_A = \mathrm{maxSpec}\, Z(A)^{\text{smooth}}}$

- strong connection between rep thy & affine geometry
- get singular locus of $Z(A)$ corresponds to irreps of intermediate dimension
- But this fact is tough to establish, done on case by case basis for many PI algs

Theorem [W-Wang-Yakimov] For A = PI 3-dim'l or 4-dim'l Sklyanin algs,

① Have that $\mathcal{A}_A = \mathrm{maxSpec}(Z(A))^{\text{smooth}}$

② Have $\forall\, m \in \mathrm{maxSpec}(Z(A))^{\text{sing}}$ the dimension & # of corresponding eq. classes of irreps of A; know $\varphi^{-1}(m)$ $m \in Y^{\text{sing}}$

see also indep. work of K. deLaet for Sklys (1707.03813)

<u>Outline for rest of talk</u> / <u>Recipe for application to other prime, f.g., Noetherian... PI algebras</u>

For A = PI 3-dim'l, 4-dim'l Sklyanin algebra

0. What are they and why care about their rep theory?

1. Presentation of center $\underbrace{Z(A)}_{=:Z}$ & singular locus of $\underbrace{\text{maxSpec } Z(A)}_{=:Y}$

2. Want $\mathcal{A}_A \supseteq Y^{smooth}$ (already have $\subseteq$): use Brown-Gordon's framework: Poisson Z-orders

    2a. Show A is a Poisson Z-order so that induced Poisson bracket on Z is non-vanishing

    2b. Dice up Y into (2-dim'l) Poisson subvarieties (0 or 2-dim'l) & then further into union of symplectic cores. Show $(Y_{slice})^{sing}$ has codim ≥ 2 in Y_{slice}

    2c. $\boxed{\text{Show } \mathcal{R}_A \text{ has codim} \geq 2 \text{ in } Y}$ by showing each $(Y_{slice})^{smooth} \subseteq \mathcal{A}_A$ (in the e.g. case)

3. Use result of Brown & geometry of "fat point modules" to draw conclusions about $\{\Phi^{-1}(\mathfrak{m})\}_{\mathfrak{m} \in Y^{sing}}$
$\subseteq$ Irreps of A of intermed. dim.

The common "step" on the way to showing $\mathcal{A}_A \supseteq Y^{smooth}$

though the routes can be different

$\mathcal{A}_A = Y^{smooth}$ due to [Brown-Goodearl 1997]

<u>0. Sklyanin algebras</u> of dim d are connected graded $\mathbb{C}$-algebras with d generators of deg 1 & $\binom{d}{2}$ quadratic relations (omitting the specific pres. here)

* they have many nice ring-theoretic & homological properties

- Noetherian, domain, Hilbert series same as that of $\mathbb{C}[x_1, \ldots, x_d]$
- Auslander/AS-regular, Gorenstein, ---

* not nice as they don't have a (convenient) PBW/monomial basis.

* Come equipped with projective geometric data that can be used to define/study them:

$$\Big(\underset{\text{elliptic curve}}{E \subseteq \mathbb{P}^{d-1}},\ \mathcal{O}_{\mathbb{P}^{d-1}}(1)|_E,\ \underset{\text{translation by a point}}{\sigma \in \mathrm{Aut}(E)}\Big)$$

* In fact, the rep-theoretic dichotomy for S is determined by $\sigma \in \mathrm{Aut}(E)$ when $d=3,4$:

<u>Theorem</u> [Artin-Tate-van den Bergh ($d=3$), Smith ($d=4$)]
S is module-finite/$Z(S)$ $\iff$ $|\sigma|<\infty$. In this case, $\boxed{\mathrm{PI\,deg}\, S = |\sigma|}$.

* Historically, the 4-dim'l Sklyanin algebras were the first to be introduced by EK Sklyanin in 1982-83. He was interested in exactly solvable models in statistical mechanics, in particular cooking up L-operators that satisfy the equation of operators $R(u-v)\,L_1(u)\,L_2(v) = L_2(v)\,L_1(u)\,R(u-v)$
where $R(u)$ is an elliptic solution (R-matrix) of the qYBE comprised of theta functions
spectral param.
amazingly S is indep of spec. param.
Constructing such L-operators $\iff$ finding reps of 4-dim'l Skly. alg.
He constructed several families of finite-dim'l representations & in 1982 deemed this algebra to be "deserving of the greatest attention of mathematicians". Many ring-theoretic & representation-theoretic questions were posed in his 1983, many of which were addressed by Stafford & Smith.

* The 3-dim'l Sklyanin algebras appeared a few years later in Artin-Schelter & Artin-Tate-van den Bergh's study of noncom. graded analogues of $\mathbb{C}[x,y,z]$; known as AS-regular algebras. They were the most interesting because they were the most difficult to handle (lack of PBW bases).
Its rep-theory also has physical implications - in string theory due to work of Berenstein-Jejjala-Leigh (2005). Namely they are important for studying marginal deformations of $N=4$ SYM theory.

1. On $Z(S)$ & Y^{sing}.

Assume $|\sigma| =: n < \infty$. $Y := \operatorname{maxSpec} Z(S)$, $S = PI\ Skly_3$ or $PI\ Skly_4$

Theorem [Smith-Tate]

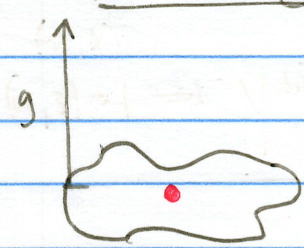
$$Z(\substack{PI \\ Skly_3}) = \frac{\mathbb{C}[z_1, z_2, z_3, g]}{(F)} \quad \& \quad Z(\substack{PI \\ Skly_4}) = \frac{\mathbb{C}[z_0, \dots, z_3, g_1, g_2]}{(F_1, F_2)}$$

z_i, g, F homogeneous of degrees $n, 3, 3n$ resp. $\qquad$ z_i, g_ℓ, F_j homog. of deg $n, 2, 2n$ resp.

$\rightsquigarrow$ $Y = \mathbb{V}(F) \subseteq \mathbb{A}^4$, dim 3 $\qquad$ both are comp. intersections $\qquad$ $Y = \mathbb{V}(F_1, F_2) \subseteq \mathbb{A}^6$, dim 4.

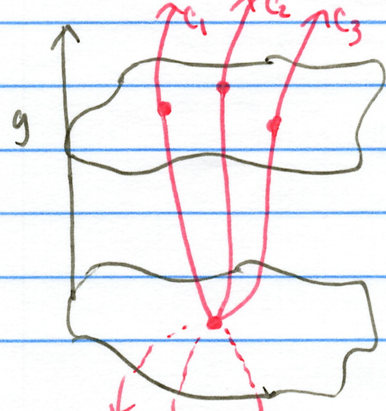
Theorem [WWY]: The singular locus of Y is given as follows.

<u>For $PI\ Skly_3$:</u>

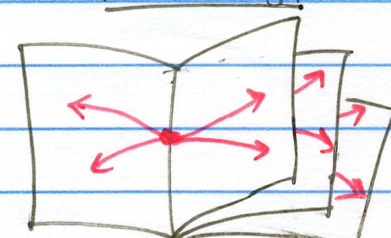


$Y^{sing} = \{\underline{0}\}$ for $(3, n) = 1$

Y^{sing} = union of 3 curves meeting at $\underline{0}$ for $3 | n$

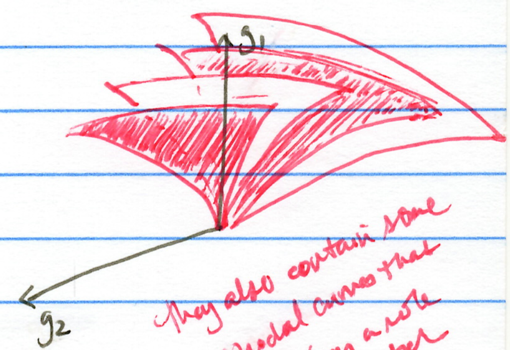


<u>For $PI\ Skly_4$</u>



Y^{sing} = union of $2(n-1)$ nodal curves meeting at $\underline{0}$ for n odd

Y^{sing} = union of 4 surfaces meeting at $\underline{0}$ for n even



They also contain some nodal curves that form a rose cluster

Recall the goal is to slice up the aff variety Y using Poisson geometry. This will help us better understand irreps esp of intermed. dim. Suppose for instance PI deg $A = 100$ there's lots of choices of what your irreps of interm. dim could be.

2. Poisson Z-orders & the Azumaya locus.
due to Brown & Gordon

A $\mathbb{C}$-algebra A that's module finite over its center Z is a Poisson Z-order if $\exists$ $\mathbb{C}$-linear map $\partial : Z \longrightarrow \mathrm{Der}_{\mathbb{C}}(A/Z)$ (space of deriv of A fixing Z)
so that the induced bracket
$$\{z, z'\} := \partial_z(z') \quad \forall z, z' \in Z$$
gives $\left\{ \begin{array}{l} Z \\ Y \end{array} \right.$ the structure of a $\left\{ \begin{array}{l} \text{Poisson alg.} \\ \text{variety.} \end{array} \right.$

If A is a Poisson Z-order, then can partition Y via "symplectic cores"
(algebraic version of symplectic leaf)
Symplectic cores are equivalence classes of $\mathfrak{m} \in Y$, where
$$\mathfrak{m} \sim \mathfrak{m}' \iff \mathcal{P}(\mathfrak{m}) = \mathcal{P}(\mathfrak{m}')$$
the unique max'l Poisson ideal $\subseteq \mathfrak{m}$

Theorem [Brown-Gordon] If $\mathfrak{m}$ & $\mathfrak{m}'$ are in the same symp. core, then $A/\mathfrak{m}A \simeq A/\mathfrak{m}'A$ as finite-dim'l algebras.

$\rightsquigarrow$ can organize irreps of A using Poisson geometry instead of just affine geometry.

2a. Theorem [WWY] Both the PI 3-dim'l, 4-dim'l Sklyanin algebras admit the structure of Poisson Z-order so that $\{,\}$ on Z is nonvanishing. Got that $\{,\}$ admits a Jacobian structure. & that g, g_1 & g_2 are in the Poisson center.

hardest part: done by modifying BG & other's method of specialization (ask after talk).

"Level N" specialization: ("Level 1" specialization was used prev. in literat.)

Given a formal deformation $A_\hbar$ of A / $\mathbb{C}[\hbar]$ or $\mathbb{C}[[\hbar]]$ so that we have a canonical projection

$$\theta: A_\hbar \twoheadrightarrow A_\hbar/(\hbar) \cong A$$

* take section $\iota: Z \hookrightarrow A_\hbar$ so that $\theta\iota = id_Z$
* define $z \in Z$: $\partial_z: Z \longrightarrow Der_{\mathbb{C}}(A/Z)$

$$z' \longmapsto \theta\left(\frac{1}{\hbar^N}[\iota(z), \tilde{z}']\right) =: \partial_z(z')$$

for $\tilde{z}' \in \theta^{-1}(z')$

$\rightsquigarrow$ A is a Poisson Z-order (of level N)

This is well-defined

Let's unpack this (towards the nonvanishing property)

① $N \geq 1$: $\tilde{z}' \in \theta^{-1}(z') \Rightarrow \theta([\iota(z), \tilde{z}']) = [z, z'] = 0$

$\Rightarrow \theta([\iota(z), \tilde{z}']) \in (\hbar)$

② We show that $\exists$ maximum value N so that S = PI 3-dim'l, PI 4-dim'l is a Poisson Z-order of level N.

③ Then we show that if ∂ vanished, we could construct a Poisson Z-order structure on such S of level $N+1$ # $\therefore$ ∂ doesn't vanish

argument involves results on derivations on S and the corresp. twisted homog. coordinate rings due to Artin-Schelter-Tate.

2b. Since $\begin{cases} g \\ g_1, g_2 \end{cases}$ are in the Poisson center Z, consider slices of Y: $\begin{cases} Y_\gamma = Y \cap \mathbb{V}(g-\gamma) \\ Y_{\gamma_1,\gamma_2} = Y \cap \mathbb{V}(g_1-\gamma_1, g_2-\gamma_2) \end{cases}$ for $\gamma, \gamma_1, \gamma_2 \in \mathbb{C}$.

These are 2-dim'l Poisson subvarieties of Y

Consider also Y_0^{symp} = subvariety of Y where $\{,\}$ vanishes.
Get $Y_0^{symp} \cap Y_{slice} = (Y_{slice})^{sing}$

In general $Y_0^{symp} = \bigcup_{slice} (Y_{slice})^{sing}$
$\cup\!|$
not nec =
though it is for 3-dim'l S & 4-dim'l S for n even
$Y^{sing} = \bigcup_{slice} (Y^{sing})_{slice}$

for 4-dim'l S with n odd $Y_0^{symp} = Y^{sing} \cup 2$ nodal curves

Y_{slice} 2-dim'l Poisson variety $\Longrightarrow$ ~~proper subvariety~~ new is 0 dim'l consists of finitely many points.

understand geometry of Y

So $(Y_{slice})^{sing}$ has codim ≥ 2 in Y_{slice}. (not a curve due to the Poisson geom.)

2c. Note that the symplectic core decomposition of Y_{slice} is
$\underset{\text{2-dim'l}}{(Y_{slice})^{smooth}} \sqcup \underset{\sqcup \text{ symp. pts.} \quad \text{0-dim'l}}{(Y_{slice})^{sing}}$

By considering a $\mathbb{C}^*$-orbit on Y (coming from rescaling gens of S) get
$\underset{\text{3-dim'l variety}}{\mathbb{C}^* \cdot (Y_{slice})^{smooth}}$ intersects nontrivially the Azumaya locus of
$\begin{cases} \text{PI 3-dim'l Sklyals } (\dim Y = 3) \\ \text{Factor of PI-dim'l Sklyal } (\dim Y = 4) \end{cases}$
due to density. Repeat argument again for PI dim'l Sklyal to conclude that $\bigcup_{slice} (Y_{slice})^{smooth} \subseteq \mathcal{A}_S$.

3. On the ireps of A.

(close to being out of time)

Use Paul Smith's analysis/geometry of fat point modules to understand the dimension and how many ireps of intermediate dimension lie over $\mathfrak{m} \in Y^{sing}$..

- A $\boxed{\text{fat pt module}}$ is a simple object in qgr A: cyclic graded module N with $H_N(t) = \frac{\varepsilon}{1-t}$, $\varepsilon = \text{mult}(N)$.
- Each irep M of A is a simple quotient of a fat point module N. In this case $\boxed{\text{mult}(N) \le \dim(M)}$. (= in nice cases)

Also use result of Braun: Fix $y \in Y$.

$$\sum_{\substack{\text{irep of } A \text{ with} \\ \text{central annih. } \mathfrak{m}_y}} \dim(\text{irep}) \le \text{PIdeg}(A)$$

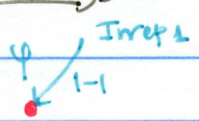
fat pt modules have multip. 1 or $n/(n,3)$

fat pt modules have multiplicity $1, \ldots, n/(n,2)$

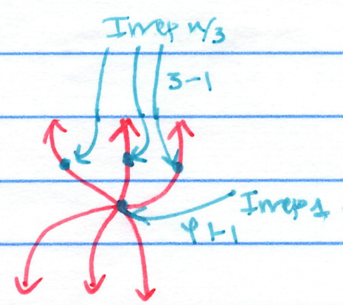
Theorem [WWY] $\varphi : \text{Irrep}(A) \longrightarrow Y$

For PISkly3

$(n,3)=1$

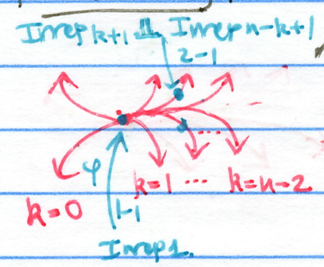


$3 | n$

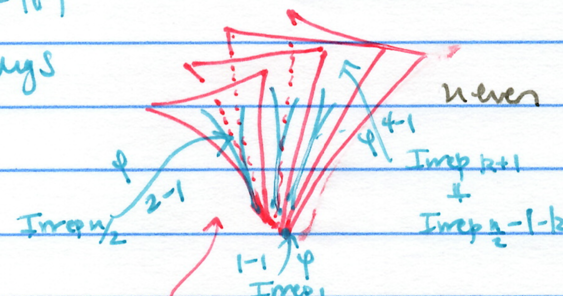


generically ireps have dim $n = |\sigma|$ = PIdeg

For PISkly4



n odd



n even

Contains some nodal curves as well (not as well understood.)