REPRESENTATION THEORY OF ELLIPTIC ALGEBRAS

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BASED ON JOINT WORK WITH
XINGTING WANG (HOWARD)
& MILEN YAKIMOV (NORTHEASTERN)

· STANFORD $\underbrace{\text{AG}}$ SEMINAR ·

... HMM, HOW IS THIS RELATED TO AG?

ALGEBRA ≡ SOLVING EQUATIONS $f(x) = 0$

x = NUMBERS

⇓

COMMUTATIVE ALG.

... SOLVE VIA
GEOMETRIC METHODS

⇓

AG

x = MATRICES

⇓

NONCOMMUTATIVE ALG.

... IN PARTICULAR

REPRESENTATION THY

GOAL: TO USE AG (& POISSON GEOMETRY) TO STUDY REPRESENTATIONS OF ALGEBRAS THAT ARE "CLOSE" TO BEING COMMUTATIVE

HERE: AN ALGEBRA IS A $\mathbb{C}$-ALGEBRA

$\equiv$ RING WITH COMPATIBLE $\mathbb{C}$-VECTOR SPACE STRUCTURE

∞-DIM'L FOR THIS WORK

Ex. POLY'L ALGEBRAS $\mathbb{C}[x_1, \dots, x_n]$

A REPRESENTATION OF AN ALGEBRA A

$\equiv$ $\mathbb{C}$-VS V EQUIPPED WITH AN ALGEBRA MAP

$$\rho : A \longrightarrow \mathrm{End}_{\mathbb{C}}(V) \cong \mathrm{Mat}_{\dim_{\mathbb{C}} V}(\mathbb{C})$$

$\equiv$ LEFT A-MODULE V OVER $\mathbb{C}$ WITH STR. MAP $A \times V \longrightarrow V$, $(a, v) \mapsto \rho(a)(v)$

ITS DEGREE / DIMENSION $\equiv$ $\dim_{\mathbb{C}} V$.

A REPRESENTATION OF AN ALGEBRA A IS IRREDUCIBLE IF $\nexists$ PROPER SUBSPACE W OF V $\ni$. $\rho(a)(w) \in W$ $\forall a \in A$, $w \in W$.

FACT: IF A IS COMMUTATIVE, THEN IRREPS OF A ARE 1-DIM'L.

REPS OF COM. ALGS ARE BORING

GOAL: TO USE AG
(& POISSON GEOMETRY)
TO STUDY REPRESENTATIONS
OF ALGEBRAS THAT ARE
"CLOSE" TO BEING COMMUTATIVE *

OUTLINE:

- PI ALGEBRAS *
- REPS OF PI ALGS USING GEOM.
- RESULTS FOR PI ELLIPTIC ALGS.

≡ PI ALGEBRAS ≡

AN ALGEBRA A SATISFIES A POLYNOMIAL IDENTITY (OR IS PI) IF $\exists$ MONIC MULTILINEAR POLYNOMIAL $f \in \mathbb{Z}\langle \underline{t} \rangle$ s.t. $f(a_1, \dots, a_n) = 0 \quad \forall a_i \in A$.

IF A IS ALSO PRIME $(aAb \neq 0 \ \forall a, b \neq 0)$ THEN ITS $\mathrm{PIdeg}(A) = \frac{1}{2}$ (MIN'L DEG OF POLY'L IDENTITY).

EXAMPLES OF PRIME PI ALGEBRAS —

A	f	$\mathrm{PIdeg}(A)$
COMMUTATIVE ALG	$t_1 t_2 - t_2 t_1$	1
$\mathrm{Mat}_d(\mathbb{C})$	$S_{2d} = \sum_{\sigma \in S_{2d}} (\mathrm{sgn}\,\sigma)\, t_{\sigma(1)} t_{\sigma(2)} \cdots t_{\sigma(2d)}$ "standard identity"	d
$\mathbb{C}\langle x, y \rangle / (xy + yx) = \mathbb{C}_{q=-1}[x, y]$	$\cdots$	2

$\mathcal{U}_q(\mathfrak{g})$ QUANTIZED ENV. ALG. ...

root of unity

$\mathcal{O}_q(G)$ QUANTIZED FUNC. ALG. ...

explicitly depends on order of q

PRIME

PI ALGEBRAS — $\mathcal{O}_q(G)$, $\mathcal{U}_q(\mathfrak{g})$, COMMUTATIVE ALGS, $\mathrm{Mat}_n(\mathbb{C})$, $\mathbb{C}\langle x,y\rangle / (xy - qyx)$ for q a root of 1

$\supseteq$ ALGEBRAS A THAT ARE MODULE FINITE OVER $Z(A)$

(ALL EXAMPLES BELONG HERE)

... PIDEG $\equiv$ MEASURE OF NONCOMMUTATIVITY

ROUGHLY: $\mathrm{RANK}_{Z(A)} A = (\mathrm{PIDEG}\ A)^2$

$A =$ COMMUTATIVE ALGEBRA	$Z(A) = A$	$\mathrm{PIDEG}(A) = 1$
$A = \mathrm{Mat}_d(\mathbb{C})$	$Z(A) = \mathbb{C}$	$\mathrm{PIDEG}(A) = d$
$A = \mathbb{C}\langle x,y\rangle / (xy+yx)$	$Z(A) = \mathbb{C}[x^2, y^2]$	$\mathrm{PIDEG}(A) = 2$

≡ IRREPS OF PI ALGEBRAS ≡

A	$\rho : A \longrightarrow Mat_n(\mathbb{C})$ IRRED.
COMMUTATIVE ALG PIDEG 1.	$\rightsquigarrow$ $n = 1$
$Mat_d(\mathbb{C})$ PIDEG d	$\rightsquigarrow$ $n = d$ $Mat_d(\mathbb{C})$ IS "AZUMAYA" ≡ ALL IRREPS HAVE SAME DIM
$\mathbb{C}\langle x,y\rangle / (xy+yx)$ PIDEG 2	$\rightsquigarrow$ $n = 1$ or 2

THEOREM (ARTIN, BROWN, GOODEARL, DE CONCINI, PROCESI, SMALL...)

IF A IS A PRIME PI ALGEBRA, FIN. GEN. & NOETHERIAN

THEN (1) DIM (IRREP OF A) $\leq$ PI DEGREE OF A

SO FIN. DIM'L.

...

≡ GEOMETRY OF IRREPS OF PI ALGEBRAS ≡

TAKE IRREP $\rho : A \longrightarrow Mat_n(\mathbb{C})$ OF A

& ITS EQUIVALENCE CLASS $[\rho]$

GET $\ker\rho$ IS A MAXIMAL IDEAL OF A

$\rightsquigarrow$ $\ker\rho \cap Z(A) =: \mathfrak{m}_\rho$ IS A MAXIMAL IDEAL OF $Z(A)$

"CENTRAL CHARACTER"

THEOREM (ARTIN, BROWN, GOODEARL, DE CONCINI, PROCESI, SMALL...)

IF A IS A PRIME PI ALGEBRA, FIN. GEN. & NOETHERIAN

THEN ① DIM(IRREP OF A) ≤ PI DEGREE OF A

② $$\begin{aligned} \mathrm{Irrep}(A) &\twoheadrightarrow \mathrm{MaxSpec}(Z(A)) =: Y \\ [\rho] &\longmapsto \mathfrak{m}_\rho := \ker\rho \cap Z(A) \end{aligned}$$

IS A WELL-DEFINED, SURJECTIVE MAP.

EXAMPLES

$A =$ COM. ALG

$\rightsquigarrow Z(A) = A$

$\rightsquigarrow n = 1$

$$\mathrm{Irrep}(A) \longrightarrow \mathrm{MaxSpec}(Z(A)) = \mathrm{MaxSpec}(A)$$

$$[\rho : A \to \mathbb{C}] \longmapsto \ker\rho = \mathfrak{m}_\rho \underset{\max}{\trianglelefteq} A$$

$\updownarrow$ 1-1 anyway

MaxSpec(A)

$A = \mathrm{Mat}_d(\mathbb{C})$

$\rightsquigarrow Z(A) = \mathbb{C}$

$\rightsquigarrow n = d$

$$\mathrm{Irrep}(A) \longrightarrow \mathrm{MaxSpec}(Z(A)) = \mathrm{MaxSpec}(\mathbb{C}) = \{0\}$$

$$[\rho : A \xrightarrow{\sim} \mathrm{Mat}_d(\mathbb{C})] \longmapsto \ker\rho = \{0\}$$

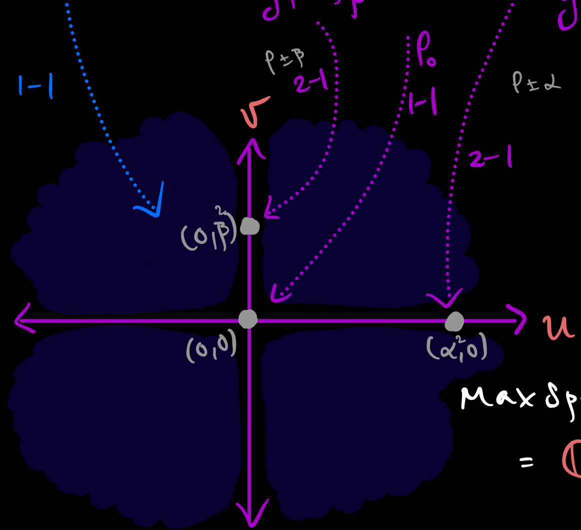
$A = \mathbb{C}\langle x,y\rangle / (xy+yx)$

$\rightsquigarrow Z(A) = \mathbb{C}[\underset{\substack{\parallel \\ u}}{x^2}, \underset{\substack{\parallel \\ v}}{y^2}]$

$\rightsquigarrow n = 1,2$

$$\boxed{\mathrm{Irrep}(A)}$$
$$\parallel$$
$$\mathrm{Irrep}^{2\text{-dim}}(A) \sqcup \mathrm{Irrep}^{1\text{-dim}}(A)$$

$$\left[\rho_\beta : \begin{array}{l} A \longrightarrow \mathbb{C} \\ x \longmapsto 0 \\ y \longmapsto \beta \end{array}\right] \qquad \left[\rho_\alpha : \begin{array}{l} A \longrightarrow \mathbb{C} \\ x \longmapsto \alpha \\ y \longmapsto 0 \end{array}\right]_{\alpha \in \mathbb{C}}$$



$\mathrm{MaxSpec}(Z(A)) = \mathbb{C}^2_{(u,v)}$

$\ker \rho_\alpha \cap \mathbb{C}[u,v]$

$= (x-\alpha, y) \cap \mathbb{C}[u,v]$

$= (u-\alpha^2, v)$

THEOREM (ARTIN, BROWN, GOODEARL, DE CONCINI, PROCESI, SMALL ...)

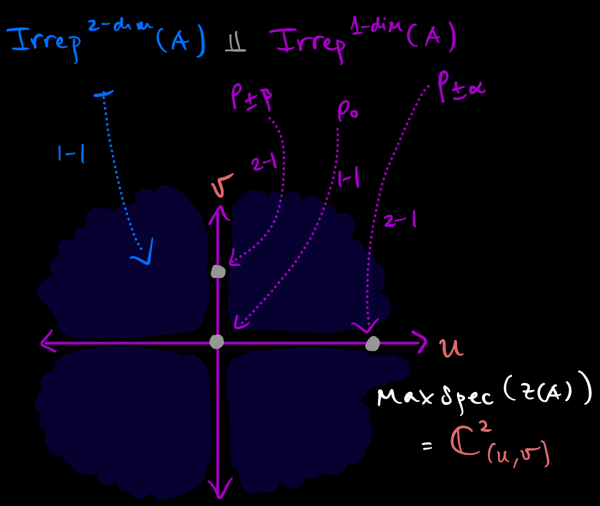
IF A IS A PRIME PI ALGEBRA, FIN. GEN. & NOETHERIAN

THEN ① DIM (IRREP OF A) $\leq$ PIDEGREE OF A

② $$\begin{aligned} \mathrm{Irrep}(A) &\twoheadrightarrow \mathrm{MaxSpec}(Z(A)) =: Y \\ [\rho] &\longmapsto \mathfrak{m}_\rho := \ker\rho \cap Z(A) \end{aligned}$$

IS A WELL-DEFINED, SURJECTIVE MAP.

③ $$\begin{aligned} \mathrm{Irrep}(A) &= \mathrm{Irrep}^{\mathrm{PIdeg}(A)}(A) \xmapsto{1-1} \mathcal{A}_A \overset{\text{open, dense}}{\subseteq} Y^{\text{SMOOTH}} \\ &\quad \text{"AZUMAYA LOCUS"} \\ &\amalg \\ &\quad \mathrm{Irrep}^{n<\mathrm{PIdeg}(A)}(A) \xmapsto{\text{finite-1}} \mathcal{R}_A \overset{\text{subvariety}}{\subseteq} Y \\ &\quad \text{"RAMIFICATION LOCUS"} \end{aligned}$$



EXAMPLE : $A = \mathbb{C}\langle x,y\rangle / (xy+yx)$

$$\mathcal{A}_A = \mathbb{C}^2_{(u,v)} \setminus \mathbb{V}(uv) \subsetneq Y^{\text{smooth}}$$

BUT IT IS CONVENIENT TO HAVE

A = PRIME PI ALG* WITH PI DEG n

$$\mathcal{A}_A = Y^{\text{smooth}}$$

AZUMAYA LOCUS

$\mathfrak{m} \in \operatorname{maxSpec} Z(A)$ s.t. $A/\mathfrak{m} \cong \operatorname{Mat}_n(\mathbb{C})$

AFFINE VARIETY $Y = \operatorname{max\,Spec} Z(A)$

$\rightsquigarrow$

$$\mathcal{R}_A = Y^{\text{sing}}$$

RAMIFICATION LOCUS (NON-AZUMAYA LOCUS)

SINGULAR VARIETY

GOAL: TO USE AG (& POISSON GEOMETRY) TO STUDY REPRESENTATIONS OF ALGEBRAS THAT ARE "CLOSE" TO BEING COMMUTATIVE* ✓

THE CONVENIENT CASE —

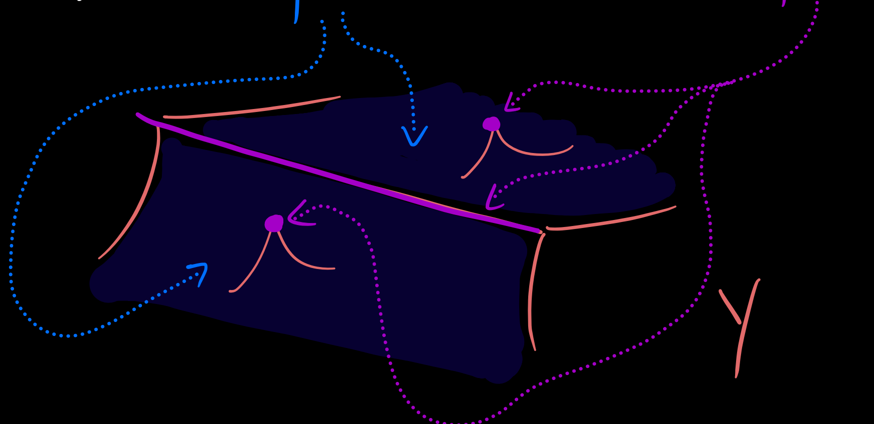
AFFINE VARIETY $Y = \operatorname{max\,Spec} Z(A)$

$\mathcal{A}_A = Y^{\text{smooth}}$

$\mathcal{R}_A = Y^{\text{sing}}$

MAX DIM IRREPS OF A $\xrightarrow{\text{1-1}}$ POINTS OF Y^{smooth}

LOWER DIM IRREPS OF A $\xrightarrow{\text{finite-1}}$ POINTS OF Y^{sing}



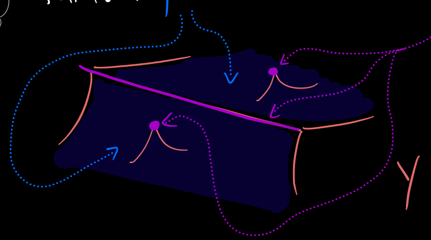
AFFINE VARIETY *
$Y = \operatorname{maxSpec} Z(A)$

$\mathcal{A}_A = Y^{smooth}$

$\mathcal{R}_A = Y^{sing}$

MAX DIM IRREPS OF A
$\downarrow$ 1-1
POINTS OF Y^{smooth}

LOWER DIM IRREPS OF A
$\downarrow$ finite-1
POINTS OF Y^{sing}



* DOESN'T HAPPEN FOR

$A = \mathbb{C}\langle x,y\rangle / (xy+yx)$ PIDEG 2, $Y = \mathbb{C}^2$

HAVE IRREPS OF DIM 1

$Y^{sing} = \emptyset$

* DOES HAPPEN FOR

$U_q(\mathfrak{g})$ QUANTIZED ENV. ALG.

root of unity

$O_q(G)$ QUANTIZED FUNC. ALG.

[BROWN - GOODEARL]

GOAL:

SHOW THAT * HAPPENS FOR MORE

PRIME PI ALGEBRAS

≡ RESULTS FOR PI ELLIPTIC ALGEBRAS ≡

CONTEXT - NONCOM. PROJ. AG...

LATE 1980'S : ARTIN, SCHELTER, TATE, VAN DEN BERGH WERE INTERESTED IN CLASSIFYING NONCOMMUTATIVE GRADED ALGEBRAS THAT RESEMBLE POLYNOMIAL ALGEBRAS

RING-THEORETICALLY & HOMOLOGICALLY

ARTIN-SCHELTER REGULAR ALGS (OF DIMENSION d)

≡ NONCOM. GRADED ALG A WITH SAME GROWTH AS $\mathbb{C}[x_1, \dots, x_d]$, GLOBAL DIM d, SATISFY A GORENSTEIN CONDITION

COMMUTATIVE
AS-REGULAR ALGS
$|||$
$\mathbb{C}[x_1, \dots, x_d]$

Ex. $\mathbb{C}\langle x, y\rangle / (xy + yx)$

& OTHER "SKEW POLY'L ALGS"

ARE AS-REGULAR

AS-REGULAR ALGS OF
DIM 1, DIM 2, DIM 3
ARE CLASSIFIED
(DIM 4+ STILL OPEN)

ONE TRICKY FAMILY
SKLYANIN ALGEBRAS
BEST STUDIED WITH GEOM. METHODS

3-DIM SKLYANIN ALG

$$S(a,b,c) = \frac{\mathbb{C}\langle x, y, z\rangle}{\begin{pmatrix} axy + byx + cz^2 \\ ayz + bzy + cx^2 \\ azx + bxz + cy^2 \end{pmatrix}}$$

$a, b, c \in \mathbb{C}$ $\ni.$ $abc \neq 0$, $(a^3+b^3+c^3)^3 \neq (3abc)^3$

$\rightsquigarrow$ EQUIPPED W/ PROJ. AG DATA TO HELP DESCRIBE BEHAVIOR.

E ELLIPTIC CURVE $\subseteq \mathbb{P}^2$
$\|$
$\mathbb{V}\left((abc)(v_1^3+v_2^3+v_3^3) - (a^3+b^3+c^3)(v_1 v_2 v_3)\right)$

σ AUTOMORPHISM OF E
$\|$
TRANSLATION BY $[a:b:c]$ $\ni.$ $O_E = [1:-1:0]$

THEOREM (ARTIN, TATE, VAN DEN BERGH, W)

CHIPPED IN A BIT

$S(a,b,c)$ IS PRIME PI $\iff$ $|\sigma| < \infty$

IN THIS CASE, $\mathrm{PIDEG}(S(a,b,c)) = |\sigma|$

Ex. $S(1,-1,0) = \mathbb{C}[x,y,z]$ COM. ALG. & $\mathrm{PIDEG} = 1 = |\sigma|$

THEOREM (SMITH–TATE) FOR $|\sigma| =: n < \infty$

PIDEG $S(a,b,c) = n$

$$Y = \operatorname{Max}\operatorname{Spec} Z(S(a,b,c)) = \mathbb{V}(F) \subseteq \mathbb{C}^4_{(u_1,u_2,u_3,g)}$$

$F = F(u_1,u_2,u_3,g)$ HOMOG. OF DEG $3n$

u_1, u_2, u_3: DEG n; g: DEG 3.

THEOREM (W–WANG–YAKIMOV)

FOR THE PI SKLYANIN ALGS $S := S(a,b,c)$, WE GET:

$$\mathcal{A}_S = Y^{\text{smooth}} \ \& \ \mathcal{R}_S = Y^{\text{sing}}$$

$$Y = \operatorname{Max}\operatorname{Spec} Z(S) = \mathbb{V}(F(u_1,u_2,u_3,g)) \subseteq \mathbb{C}^4_{(u_1,u_2,u_3,g)}$$

$\boxed{\text{PIDEG } S = |\sigma| \text{ COPRIME TO } 3}$

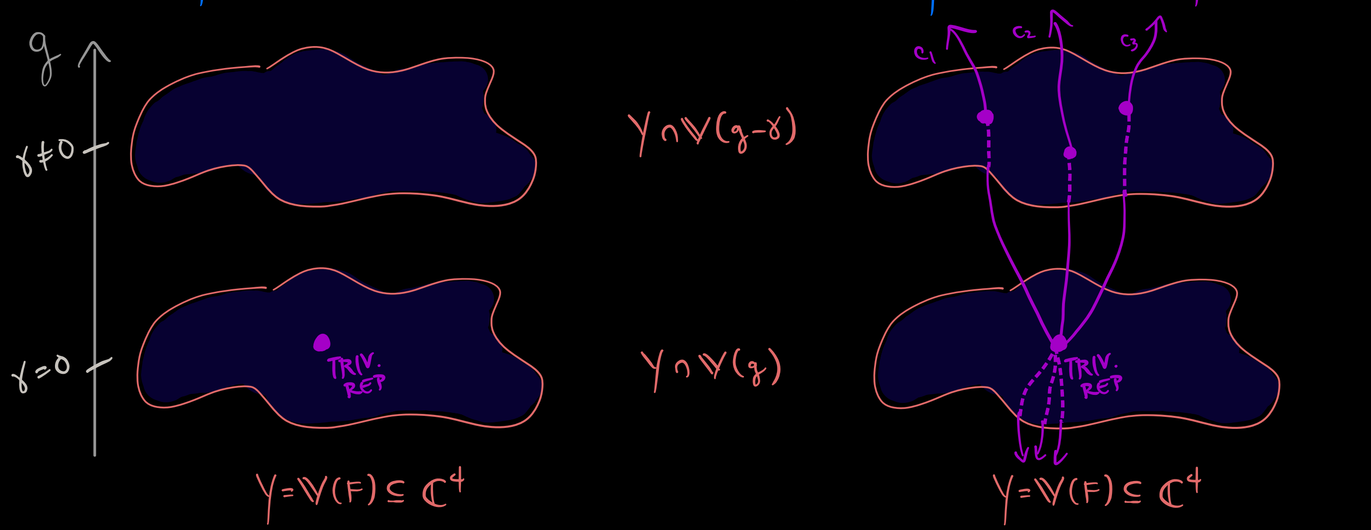
MAX DIM $(=|\sigma|)$ IRREPS OF A $\xrightarrow{1-1}$ POINTS OF Y^{smooth}

LOWER DIM $(=1)$ IRREPS OF A $\xrightarrow{1-1}$ POINTS OF $Y^{\text{sing}} = \{0\}$

$\boxed{\text{PIDEG } S = |\sigma| \text{ DIV. BY } 3}$

MAX DIM $(=|\sigma|)$ IRREPS OF A $\xrightarrow{1-1}$ POINTS OF Y^{smooth}

LOWER DIM $= \left(\frac{|\sigma|}{3} \text{ IF NON TRIV}\right)$ IRREPS OF A $\xrightarrow{3-1 \text{ IF NON TRIV.}}$ POINTS OF $Y^{\text{sing}} = C_1 \cup C_2 \cup C_3$



<u>PROOF STRATEGY</u>: $\mathcal{A}_S \subseteq Y^{\text{smooth}}$ ALREADY

TO GET $Y^{\text{smooth}} \subseteq \mathcal{A}_S$: DICE UP Y INTO <u>SURFACES</u>

$$Y = \coprod_{\gamma \in \mathbb{C}} Y_\gamma \quad \text{FOR} \quad Y_\gamma := Y \cap \mathbb{V}(g - \gamma)$$

$\rightsquigarrow$ CAN GIVE EACH SLICE Y_γ THE STRUCTURE OF A POISSON VARIETY ... BUYS US MORE REP THY TOOLS

<u>THEOREM</u> (BROWN–GORDAN) $Y_\gamma = \coprod$ "SYMPLECTIC CORES" & $\mathfrak{m}, \mathfrak{m}' \in$ SAME SYMP. CORE $\Rightarrow S/\mathfrak{m} \simeq S/\mathfrak{m}'$ AS FIN. DIM ALGS

($\mathfrak{m}, \mathfrak{m}'$ CORRESP. TO IRREPS OF THE SAME DIM)

$\rightsquigarrow Y_\gamma^{\text{smooth}} \subseteq \mathcal{A}_S \;\; \forall \gamma \in \mathbb{C} \quad \rightsquigarrow \quad Y^{\text{smooth}} \subseteq \mathcal{A}_S.$ ///

$\boxed{\text{PI DEG } S = |\sigma| \text{ COPRIME TO } 3}$

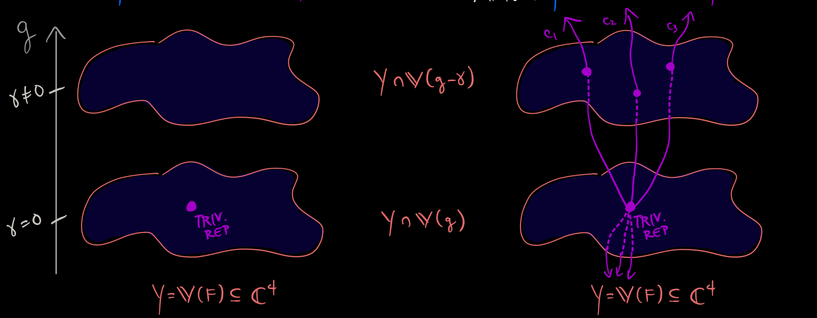
MAX DIM $(=|\sigma|)$ IRREPS OF A $\xrightarrow{1-1}$ POINTS OF Y^{smooth}

LOWER DIM $(=1)$ IRREPS OF A $\xrightarrow{1-1}$ POINTS OF $Y^{\text{sing}} = \{0\}$

$\boxed{\text{PI DEG } S = |\sigma| \text{ DIV. BY } 3}$

MAX DIM $(=|\sigma|)$ IRREPS OF A $\xrightarrow{1-1}$ POINTS OF Y^{smooth}

LOWER DIM $= \left(\frac{|\sigma|}{3} \text{ IF NON TRIV}\right)$ IRREPS OF A $\xrightarrow{3-1 \text{ IF NON TRIV.}}$ POINTS OF $Y^{\text{sing}} = C_1 \cup C_2 \cup C_3$



USED AG & POISSON GEOM. TO ORGANIZE IRREPS OF THE PI ELLIPTIC ALG $S(a,b,c)$

3-DIM SKLYANIN ALG

SUCH REPS ARE USED IN STRING THEORY

THE REPS THAT REALLY LAUNCHED THIS DIRECTION WERE THOSE OF THE

S 4-DIM SKLYANIN ALG $\leftrightarrow$ $E \subseteq \mathbb{P}^3$ $\sigma \in \mathrm{Aut}(E)$ GEOMETRIC DATA

USED IN THE "QUANTUM INVERSE SCATTERING METHOD" IN QUANTUM PHYSICS.

THEOREM (W-WANG-YAKIMOV)

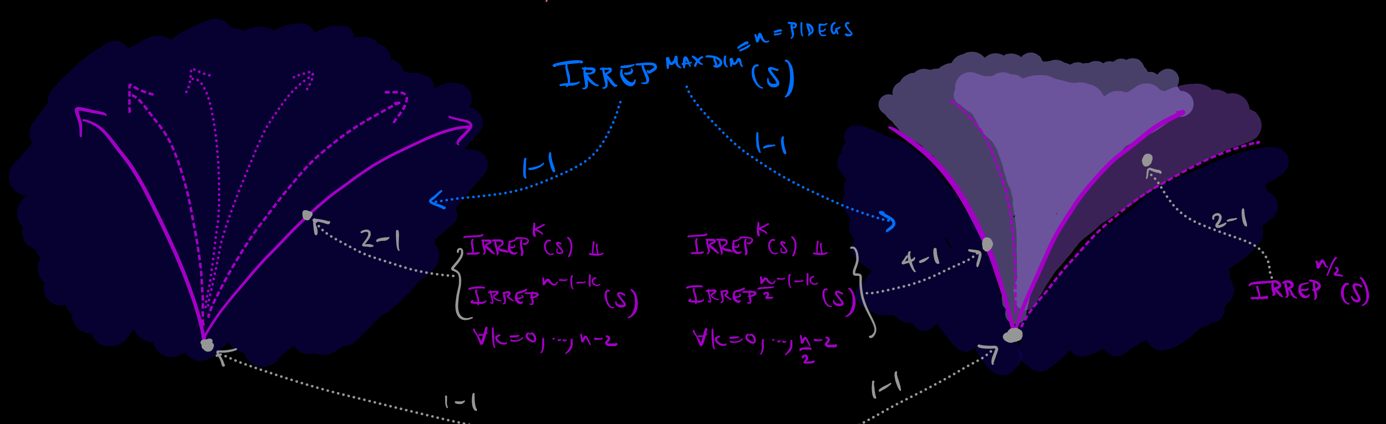
WE HAVE THAT:

$$\mathcal{A}_S = Y^{smooth}$$

$$\& \quad \mathcal{R}_S = Y^{sing}$$

FOR S PI 4-DIM SKLYANIN ALG

$\boxed{\text{PI DEG}(S) \text{ ODD}}$ $\quad Y = \mathbb{V}(F_1, F_2) \subseteq \mathbb{C}^6$ $\quad \boxed{\text{PI DEG}(S) \text{ EVEN}}$



Y^{sing} = UNION OF $2(n-1)$ CUSPIDAL CURVES

TRIV IRREP OF DIM 1

Y^{sing} = UNION OF SUBVARIETIES CONTAINING CUSP. CURVES

UPSHOT —

IF GIVEN A NICE NONCOM. ALG A
THAT'S CLOSE TO BEING COMMUTATIVE
WHERE $Z(A)$ IS KNOWN

THEN WE GET INFO
ABOUT ITS REP THY

USING GEOMETRIC TECHNIQUES.

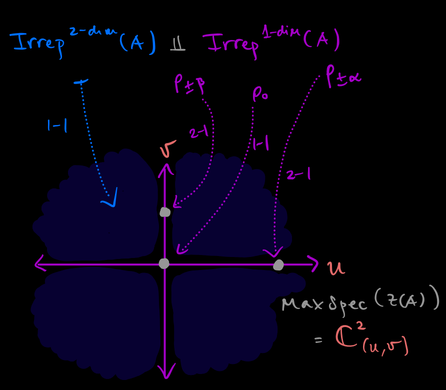
PRIME

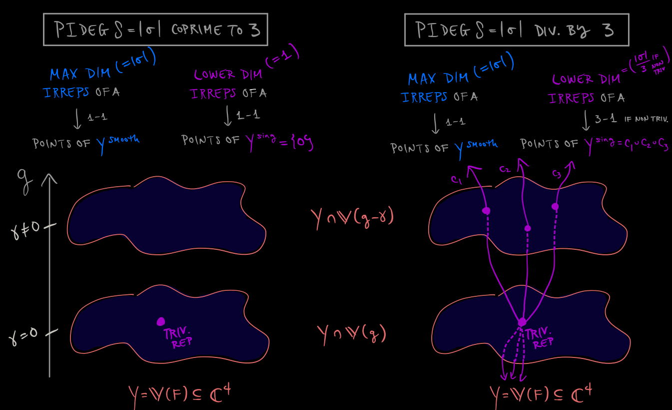
MODULE-FINITE OVER $Z(A)$
OR IS PI

FOR $Y = \mathrm{MaxSpec}\, Z(A)$
KNOW
Y^{smooth} & Y^{sing}

DIMENSIONS OF
IRREPS

ORGANIZING
IRREPS ρ VIA
"CENTRAL ANNIHILATORS"
$\mathfrak{m}_\rho \in Y$





Thanks for listening!

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