MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #6

LAST TIME

- SIMPLE ALGS.
- SEMISIMPLE ALGS.
- SEPARABLE ALGS.

TOPICS:

I. CATEGORIES (§2.1)

II. UNIVERSAL CONSTRUCTIONS (§2.2.1)

I. CATEGORIES

"CATEGORY THEORY IS THE MATHEMATICS OF MATHEMATICS."

— PROF. EUGENIA CHENG

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A CATEGORY $\mathcal{C}$ CONSISTS OF THE DATA:

(a) A COLLECTION OF OBJECTS $\mathrm{Ob}(\mathcal{C})$.

WRITE $X \in \mathcal{C}$ FOR $X \in \mathrm{Ob}(\mathcal{C})$.

(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathcal{C}$,

A COLLECTION OF MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

WRITE $g : X \to Y$ FOR $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

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WRITE $g : X \to Y$ FOR $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

(c) FOR EVERY OBJECT $X \in \mathcal{C}$,

AN IDENTITY MORPHISM $\mathrm{id}_X : X \to X$.

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A COLLECTION OF MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X, Y)$.

WRITE $g : X \to Y$ FOR $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$.

(c) FOR EVERY OBJECT $X \in \mathcal{C}$,

AN IDENTITY MORPHISM $\mathrm{id}_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS

$f : W \to X$ AND $g : X \to Y$,

A COMPOSITE MORPHISM $gf := g \circ f : W \to Y$.

I. CATEGORIES

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A **CATEGORY** $\mathcal{C}$ CONSISTS OF THE DATA:

(a) A COLLECTION OF **OBJECTS** $\mathrm{Ob}(\mathcal{C})$.
WRITE $X \in \mathcal{C}$ FOR $X \in \mathrm{Ob}(\mathcal{C})$.

(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathcal{C}$,
A COLLECTION OF **MORPHISMS** $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.
WRITE $g: X \to Y$ FOR $g \in \mathrm{Hom}_{\mathcal{C}}(X,Y)$.

(c) FOR EVERY OBJECT $X \in \mathcal{C}$,
AN **IDENTITY MORPHISM** $\mathrm{id}_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS
$f: W \to X$ AND $g: X \to Y$,
A **COMPOSITE MORPHISM** $gf := g \circ f : W \to Y$.

THIS DATA MUST SATISFY THE AXIOMS:

ASSOCIATIVITY

$$(hg)f = h(gf) \quad \text{IN} \quad \mathrm{Hom}_{\mathcal{C}}(W,Z)$$

UNITALITY

$$\mathrm{id}_X f = f \text{ IN } \mathrm{Hom}_{\mathcal{C}}(W,X) \quad \& \quad g\,\mathrm{id}_X = g \text{ IN } \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

$\forall\, f: W \to X,\ g: X \to Y,\ h: Y \to Z$

I. CATEGORIES

USE "COLLECTION" INSTEAD OF "SET"

(... TO AVOID ISSUES WITH "A SET OF SETS" LATER)

DOESN'T EXIST

A CATEGORY $\mathcal{C}$ CONSISTS OF THE DATA:

(a) A COLLECTION OF OBJECTS $\mathrm{Ob}(\mathcal{C})$.

(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathcal{C}$,
A COLLECTION OF MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.

(c) FOR EVERY OBJECT $X \in \mathcal{C}$,
AN IDENTITY MORPHISM $\mathrm{id}_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS
$f : W \to X$ AND $g : X \to Y$,
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$\forall\, f : W \to X,\ g : X \to Y,\ h : Y \to Z$

I. CATEGORIES

USE "COLLECTION" INSTEAD OF "SET"

A SMALL CATEGORY $\mathcal{C}$ CONSISTS OF THE DATA:

(a) A COLLECTION OF OBJECTS $\mathrm{Ob}(\mathcal{C})$.

(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathcal{C}$,
A COLLECTION OF MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.
& ALL MORPHISMS $\mathrm{Hom}(\mathcal{C})$ FORM A SET

(c) FOR EVERY OBJECT $X \in \mathcal{C}$,
AN IDENTITY MORPHISM $\mathrm{id}_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS
$f : W \to X$ AND $g : X \to Y$,
A COMPOSITE MORPHISM $gf := g \circ f : W \to Y$.

SPECIAL CASES

THIS DATA MUST SATISFY THE AXIOMS:

ASSOCIATIVITY

$$\underset{\substack{\text{IN} \\ \mathrm{Hom}_{\mathcal{C}}(W,Z)}}{(hg)f = h(gf)}$$

UNITALITY

$$\underset{\substack{\text{IN} \\ \mathrm{Hom}_{\mathcal{C}}(W,X)}}{\mathrm{id}_X f = f} \quad \& \quad \underset{\substack{\text{IN} \\ \mathrm{Hom}_{\mathcal{C}}(X,Y)}}{g\,\mathrm{id}_X = g}$$

$\forall\, f : W \to X,\ g : X \to Y,\ h : Y \to Z$

I. CATEGORIES

USE "COLLECTION" INSTEAD OF "SET"

A LOCALLY SMALL CATEGORY $\mathcal{C}$ CONSISTS OF THE DATA:

(a) A COLLECTION OF OBJECTS $\mathrm{Ob}(\mathcal{C})$.

(b) FOR EVERY PAIR OF OBJECTS $X, Y \in \mathcal{C}$,
A COLLECTION OF MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$.
& THIS IS A SET $\forall X, Y \in \mathcal{C}$.

(c) FOR EVERY OBJECT $X \in \mathcal{C}$,
AN IDENTITY MORPHISM $\mathrm{id}_X : X \to X$.

(d) FOR EVERY PAIR OF MORPHISMS
$f : W \to X$ AND $g : X \to Y$,
A COMPOSITE MORPHISM $gf := g \circ f : W \to Y$.

SPECIAL CASES

THIS DATA MUST SATISFY THE AXIOMS:

ASSOCIATIVITY

$$(hg)f = h(gf) \quad \text{IN} \quad \mathrm{Hom}_{\mathcal{C}}(W,Z)$$

UNITALITY

$$\mathrm{id}_X f = f \text{ IN } \mathrm{Hom}_{\mathcal{C}}(W,X) \quad \& \quad g\,\mathrm{id}_X = g \text{ IN } \mathrm{Hom}_{\mathcal{C}}(X,Y)$$

$\forall\, f : W \to X,\ g : X \to Y,\ h : Y \to Z$

I. CATEGORIES

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$\mathrm{id}_X f = f$, $g\,\mathrm{id}_X = g$

ALGEBRAIC CATEGORIES

I. CATEGORIES

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Group

GROUPS & GROUP HOMOMS.

ALGEBRAIC CATEGORIES

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Ab

ABELIAN GROUPS & GROUP HOMOMS.

NOT "ABELIAN GROUP HOMOMS"

PROPERTY

ALGEBRAIC CATEGORIES

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PROPERTY

Ring
UNITAL RINGS & UNITAL RING HOMOMS.

Rng
RINGS & RING HOMOMS.

ALGEBRAIC CATEGORIES

ComRing
UNITAL COMMUTATIVE RINGS & UNITAL RING HOMOMS

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A CATEGORY $\mathcal{C}$ CONSISTS OF:

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SATISFYING

ASSOCIATIVITY

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$\mathrm{id}_X f = f$, $g\,\mathrm{id}_X = g$

$\Bbbk$ FIELD
ALG. CLOSED
& CHAR. 0
(NOT NEEDED HERE)

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ALGEBRAIC CATEGORIES

ComRing
UNITAL COMMUTATIVE RINGS & UNITAL RING HOMOMS

Vec
$\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

FdVec
FINITE DIM'L $\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

I. CATEGORIES

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COMMUTATIVE $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

Vec
$\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

FdAlg
FINITE DIM'L $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

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A, B $\Bbbk$-ALGS

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Rep(A)
REP'NS OF A & REP'N MORPHISMS

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RINGS & RING HOMOMS.

A-Mod
Mod-A
(A,B)-Bimod
A-Bimod
VARIOUS CATEGORIES OF (BI)MODULES

ALGEBRAIC CATEGORIES

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SATISFYING

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ALGEBRAIC CATEGORIES

Group — GROUPS & GROUP HOMOMS.

Ab — ABELIAN GROUPS & GROUP HOMOMS. NOT "ABELIAN GROUP HOMOMS" ← PROPERTY

Ring — UNITAL RINGS & UNITAL RING HOMOMS.

Rng — RINGS & RING HOMOMS.

Rep(A) — REP'NS OF A & REP'N MORPHISMS

$\Bbbk$ FIELD, ALG. CLOSED & CHAR. 0 (NOT NEEDED HERE)

A, B, C $\Bbbk$-ALGS

A-Mod, **Mod-A**, **(A,B)-Bimod**, **A-Bimod** — VARIOUS CATEGORIES OF (BI)MODULES

ComRing — UNITAL COMMUTATIVE RINGS & UNITAL RING HOMOMS

Alg — $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

ComAlg — COMMUTATIVE $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

Vec — $\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

Bim

OBJECTS $\equiv$ $\Bbbk$-ALGEBRAS

MORPHISMS: $A \to B \Leftrightarrow [{}_A V_B \in (A,B)\text{-Bimod}]$ ~ISOCLASS~

WITH: $\mathrm{id}_A \equiv$???

$A \to B \to C \equiv$???

Fd Alg — FINITE DIM'L $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

FdVec — FINITE DIM'L $\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

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SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

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A, B, C $\Bbbk$-ALGS

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$\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

Bim
OBJECTS $\equiv$ $\Bbbk$-ALGEBRAS
MORPHISMS:
$A \to B \Leftrightarrow [{}_A V_B \in (A,B)\text{-Bimod}]$ ~ISOCLASS~
WITH: $\mathrm{id}_A \equiv {}_A(A_{reg})_A$
$A \to B \to C \equiv {}_A V_B \otimes_B {}_B W_C \in (A,C)\text{-Bimod}$

FdAlg
FINITE DIM'L $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

FdVec
FINITE DIM'L $\Bbbk$-VECTOR SPACES & $\Bbbk$-LINEAR MAPS

I. CATEGORIES

A **CATEGORY** $\mathcal{C}$ CONSISTS OF:

(a) **OBJECTS.**

(b) **MORPHISMS** $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

(c) $\mathrm{id}_X : X \to X$ $\forall X \in \mathcal{C}$.

(d) $gf : W \to Y$ $\forall f : W \to X$, $g : X \to Y$.

SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$\mathrm{id}_X f = f$, $g\,\mathrm{id}_X = g$

$\Bbbk$ FIELD ALG. CLOSED & CHAR. 0 (NOT NEEDED HERE)

A, B, C $\Bbbk$-ALGS

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Fd Alg
FINITE DIM'L $\Bbbk$-ALGEBRAS & $\Bbbk$-ALGEBRA HOMOMS.

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I. CATEGORIES

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

(b) MORPHISMS $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ $\forall X,Y \in \mathcal{C}$.

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SATISFYING

ASSOCIATIVITY

$(hg)f = h(gf)$

UNITALITY

$\mathrm{id}_X f = f$, $g\,\mathrm{id}_X = g$

... MORE (NON-ALGEBRAIC) EXAMPLES LATER

LET'S STUDY MORPHISMS IN DETAIL...

$$g : X \longrightarrow Y$$

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LET'S STUDY MORPHISMS IN DETAIL...

$$g : X \longrightarrow Y$$

DOMAIN OF g

CODOMAIN OF g

I. CATEGORIES

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LET'S STUDY MORPHISMS IN DETAIL...

$$g : X \longrightarrow Y$$

DOMAIN OF g — CODOMAIN OF g

g IS MONIC (OR IS A MONO) IF IT IS LEFT-CANCELLATIVE:

$\forall f, f' : W \to X$ WITH $gf = gf'$ WE GET $f = f'$.

I. CATEGORIES

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$\forall f, f' : W \to X$ WITH $gf = gf'$ WE GET $f = f'$.

HERE: $X := (X, g)$ IS A SUBOBJECT OF Y

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$\forall f, f' : W \to X$ WITH $gf = gf'$ WE GET $f = f'$.

HERE: $X := (X, g)$ IS A SUBOBJECT OF Y

g IS EPIC (OR IS AN EPI) IF IT IS RIGHT-CANCELLATIVE:

$\forall h, h' : Y \to Z$ WITH $hg = h'g$ WE GET $h = h'$.

HERE: $Y := (Y, g)$ IS A QUOTIENT OBJECT OF X

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SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

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LET'S STUDY MORPHISMS IN DETAIL...

$$g : X \longrightarrow Y$$

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$\forall f, f' : W \to X$ WITH $gf = gf'$ WE GET $f = f'$.

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EXERCISE 2.2

MONO IN Ab $\equiv$ INJECTIVE GROUP HOMOM.

EPI IN Ab $\equiv$ SURJECTIVE GROUP HOMOM.

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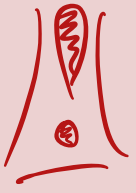

ISO $\neq$ MONO + EPI IN GENERAL

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EXERCISE 2.2

$\mathbb{Z} \hookrightarrow \mathbb{Q}$ IN Ring IS MONIC & EPIC YET IS NOT AN ISO.

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EXERCISE 2.1 (1) SHOW: g ISO $\Longrightarrow$ g MONIC & EPIC. Yoneda!

(2) $g : X \to Y$ IS $\begin{cases} \text{SPLIT-MONIC IF } \exists h : Y \to X \text{ s.t. } hg = \mathrm{id}_X. \\ \text{SPLIT-EPIC IF } \exists f : Y \to X \text{ s.t. } gf = \mathrm{id}_Y. \end{cases}$

SHOW: g SPLIT-MONIC EPI (OR SPLIT-EPIC MONO) $\Longrightarrow$ g ISO.

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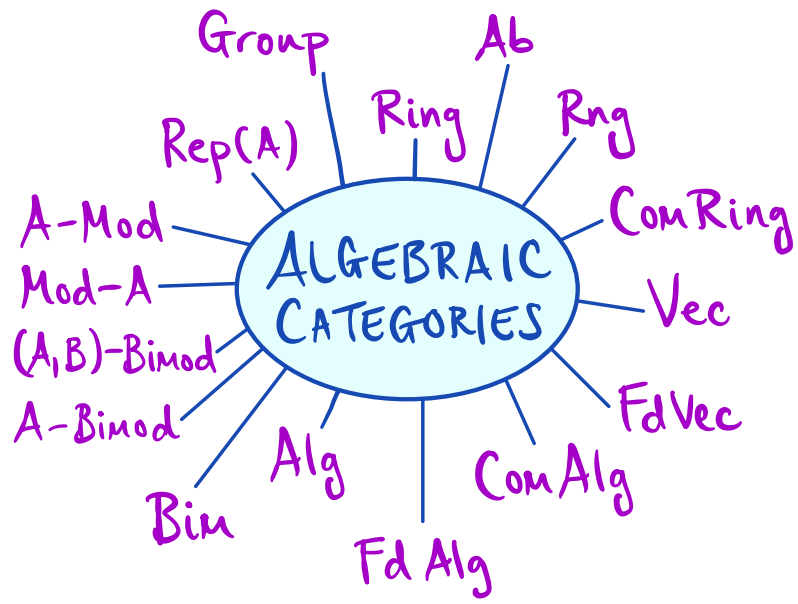

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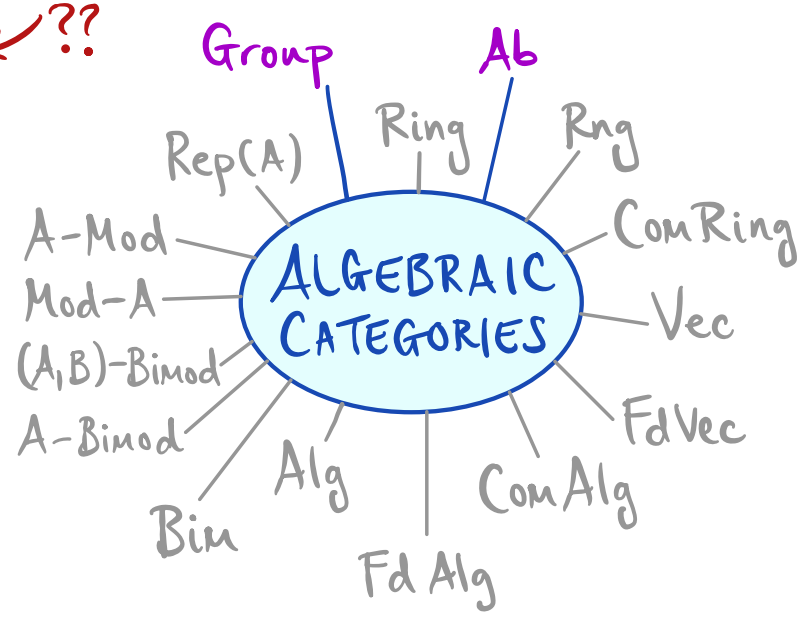

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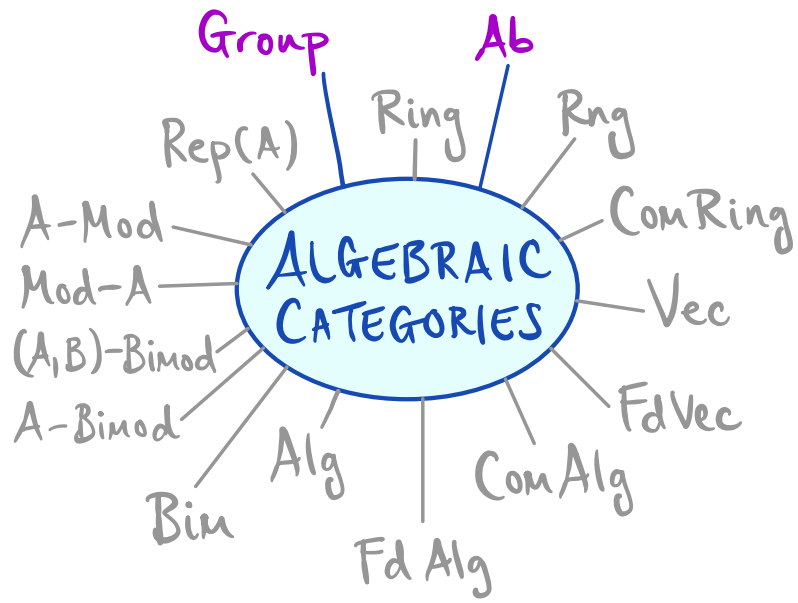

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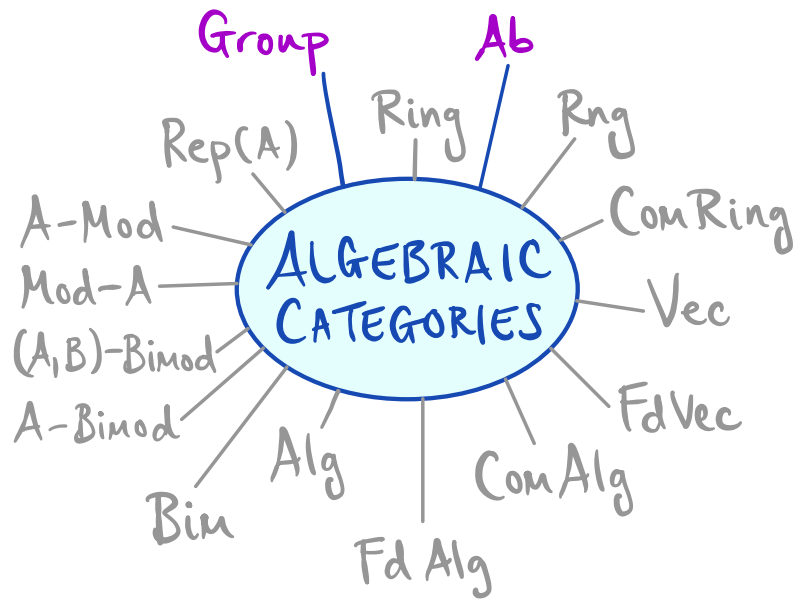
- $f \in \mathrm{Hom}(\mathcal{D}) \Rightarrow \mathrm{dom}(f), \mathrm{codom}(f) \in \mathrm{Ob}(\mathcal{D})$.
- $f, g \in \mathrm{Hom}(\mathcal{D})$ WITH $\mathrm{codom}(f) = \mathrm{dom}(g) \Rightarrow gf \in \mathrm{Hom}(\mathcal{D})$.

SUBCAT $\mathcal{D}$ OF $\mathcal{C}$ IS FULL IF

$$\mathrm{Hom}_{\mathcal{D}}(X,Y) = \mathrm{Hom}_{\mathcal{C}}(X,Y) \quad \forall X,Y \in \mathcal{D}.$$



Ab $\equiv$ SUBCATEGORY OF Group

FULL BECAUSE $\forall G, G' \in \mathrm{Ab}$:

$f \in \mathrm{Hom}_{\mathrm{Ab}}(G,G')$ IS A GROUP HOMOM.

SO $f \in \mathrm{Hom}_{\mathrm{Group}}(G,G')$

& VICE VERSA.

I. CATEGORIES

A CATEGORY $\mathcal{C}$ CONSISTS OF:

(a) OBJECTS.

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SATISFYING

ASSOCIATIVITY $(hg)f = h(gf)$

UNITALITY $\mathrm{id}_X f = f$, $g\,\mathrm{id}_X = g$

..... LET'S CHECK OUT SUBSTRUCTURES & EXAMPLES

A SUBCATEGORY $\mathcal{D}$ OF $\mathcal{C}$ CONSISTS OF:

(a) SUBCOLLECTION $\mathrm{Ob}(\mathcal{D})$ OF $\mathrm{Ob}(\mathcal{C})$.

(b) SUBCOLLECTION $\mathrm{Hom}(\mathcal{D})$ OF $\mathrm{Hom}(\mathcal{C})$.

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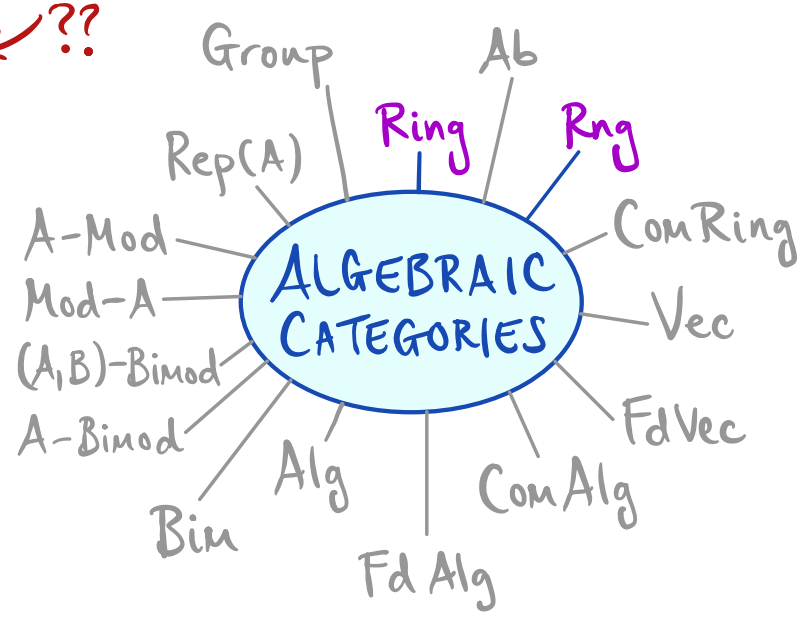
??



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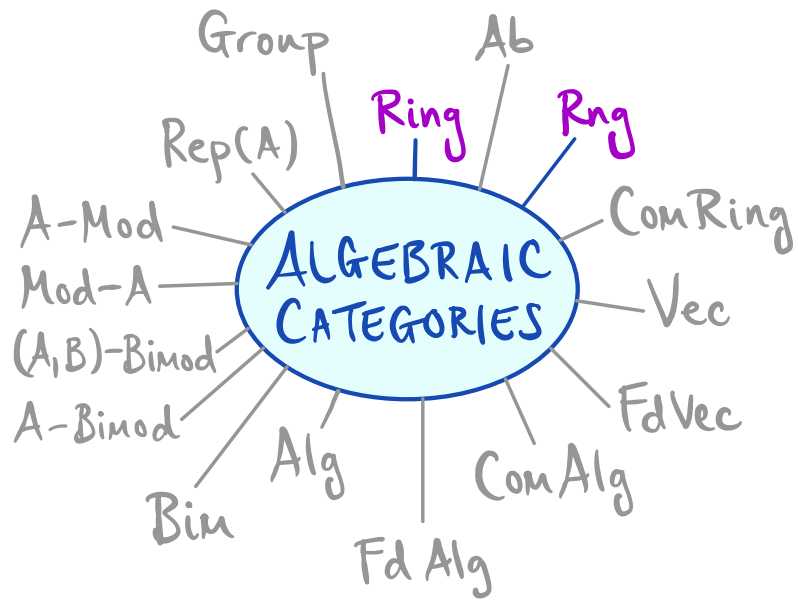

Ring $\equiv$ SUBCATEGORY OF Rng

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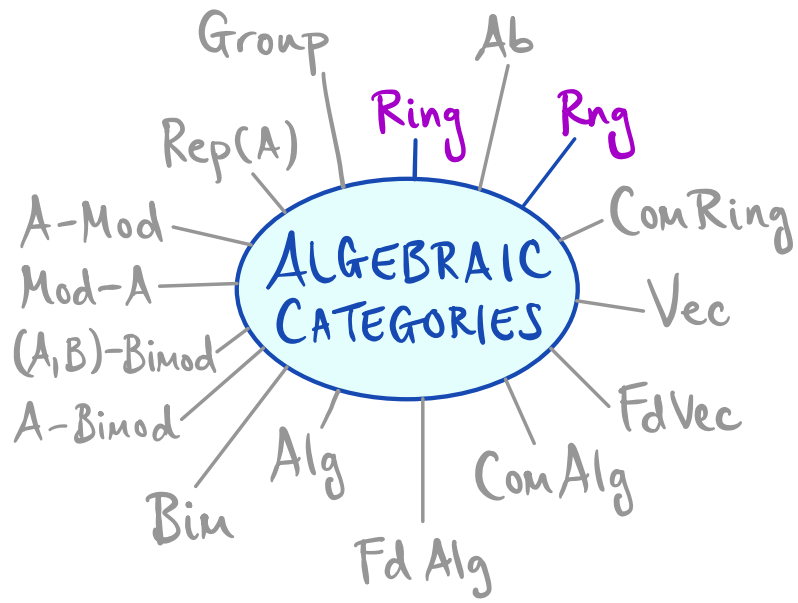

Ring $\equiv$ SUBCATEGORY OF Rng

NOT FULL BECAUSE $\forall R, R' \in \mathrm{Ring}$:

$f \in \mathrm{Hom}_{\mathrm{Rng}}(R,R')$ IS A RING HOMOM.

BUT IT DOESN'T NEED TO BE UNITAL

$\therefore \mathrm{Hom}_{\mathrm{Rng}}(R,R') \neq \mathrm{Hom}_{\mathrm{Ring}}(R,R')$.

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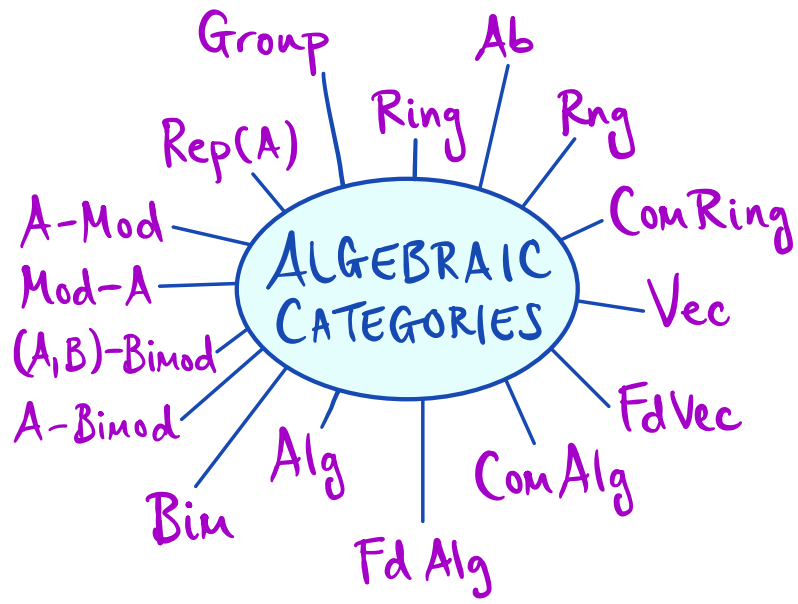
.....MORE EXAMPLES



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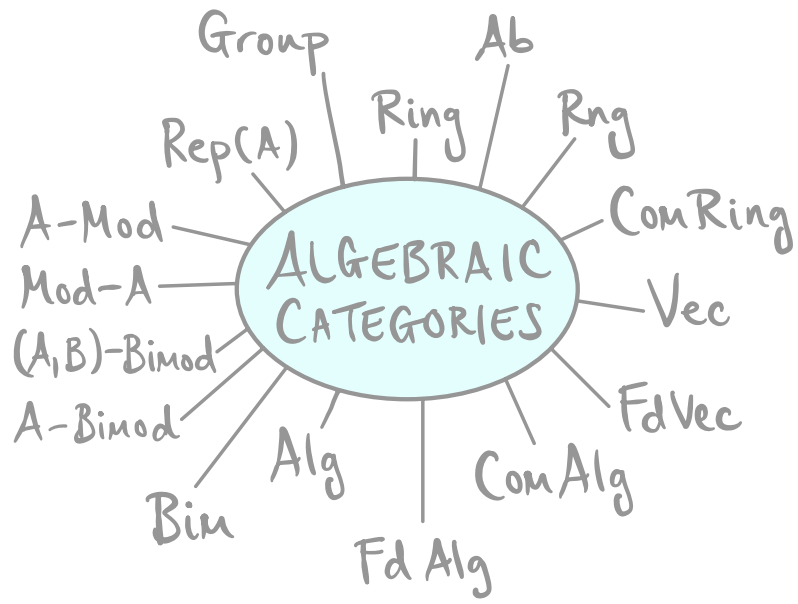
.....MORE EXAMPLES

LOGICAL/ CATEGORICAL CATEGORIES



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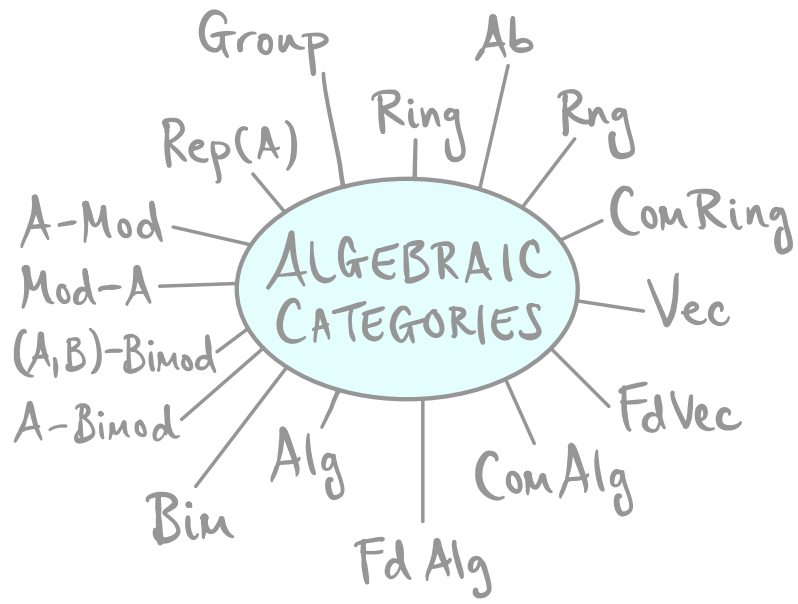
.....MORE EXAMPLES

$\emptyset$

NO OBJECTS
NO MORPHISMS

LOGICAL/ CATEGORICAL CATEGORIES



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.....MORE EXAMPLES

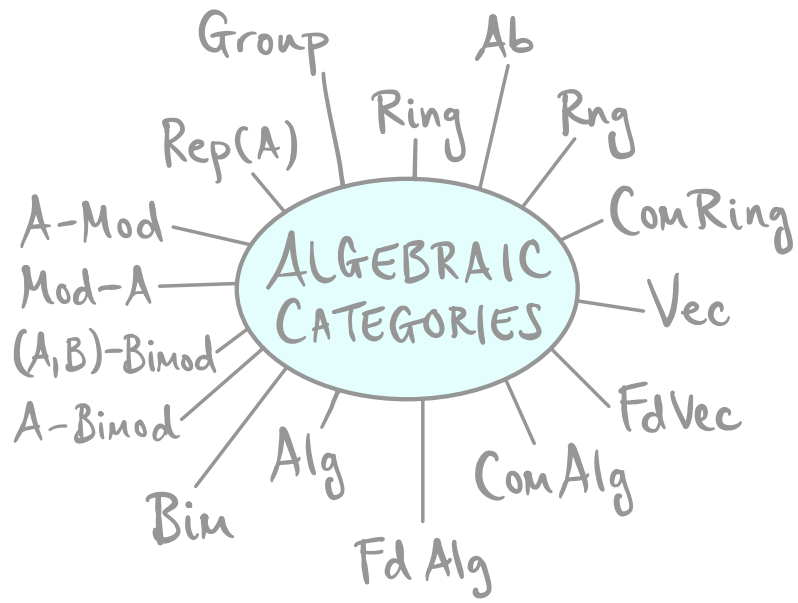
$\emptyset$

NO OBJECTS
NO MORPHISMS

Set

SETS
FUNCTIONS

LOGICAL/ CATEGORICAL CATEGORIES



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.....MORE EXAMPLES

$\emptyset$

NO OBJECTS
NO MORPHISMS

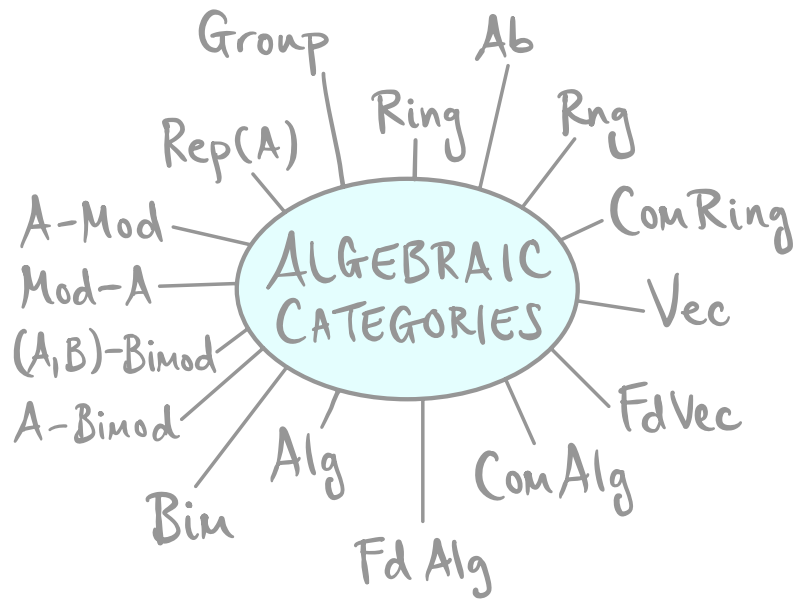
Set

SETS
FUNCTIONS

LOGICAL/ CATEGORICAL CATEGORIES

Cat

SMALL CATEGORIES
"FUNCTORS" ← LECTURE 8



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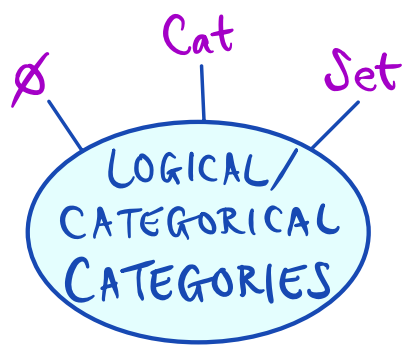

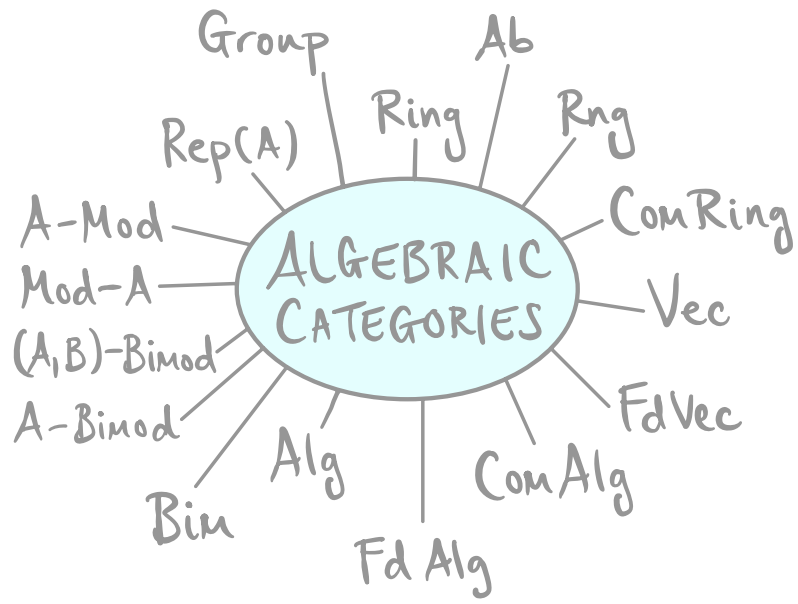

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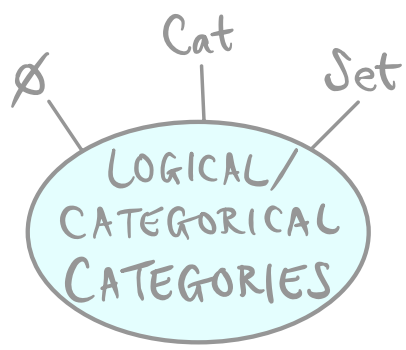

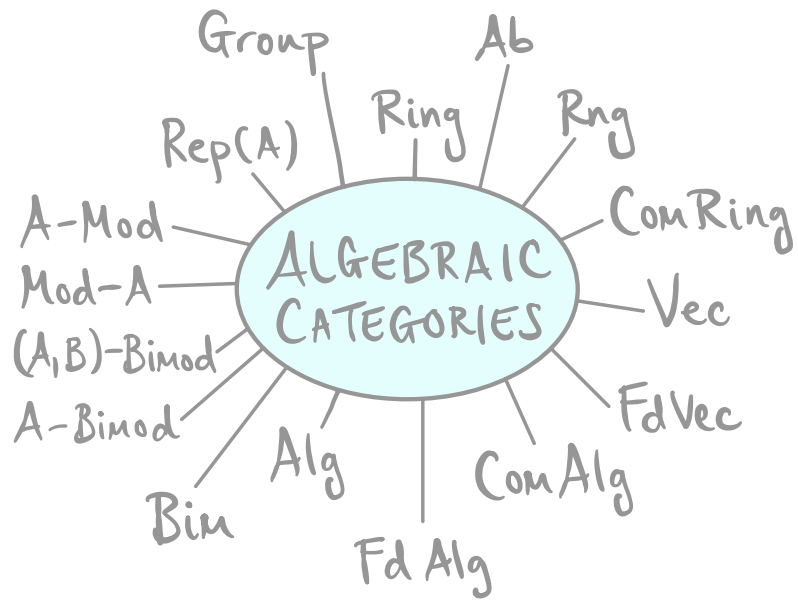

GEOMETRIC/ TOPOLOGICAL CATEGORIES

COMBINATORIAL CATEGORIES

ANALYTIC CATEGORIES

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.....MORE EXAMPLES

LOGICAL/CATEGORICAL CATEGORIES: $\emptyset$, Cat, Set

ALGEBRAIC CATEGORIES: Group, Ab, Ring, Rng, ComRing, Vec, FdVec, ComAlg, Fd Alg, Alg, Bim, A-Bimod, (A,B)-Bimod, Mod-A, A-Mod, Rep(A)

Aff { AFFINE VARIETIES, REGULAR MAPS

GEOMETRIC/TOPOLOGICAL CATEGORIES

Top { TOPOLOGICAL SPACES, CONTINUOUS MAPS

ANALYTIC CATEGORIES

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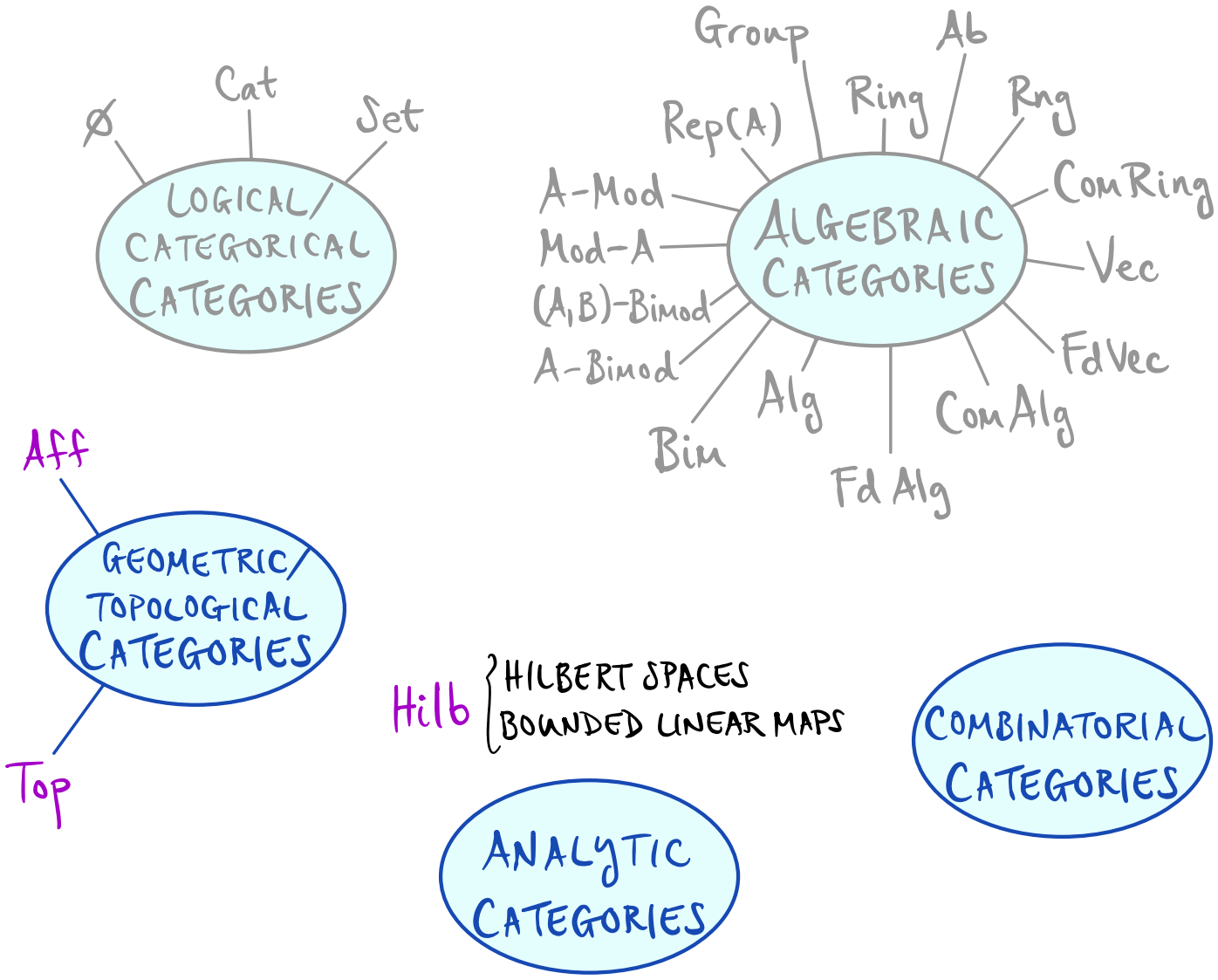

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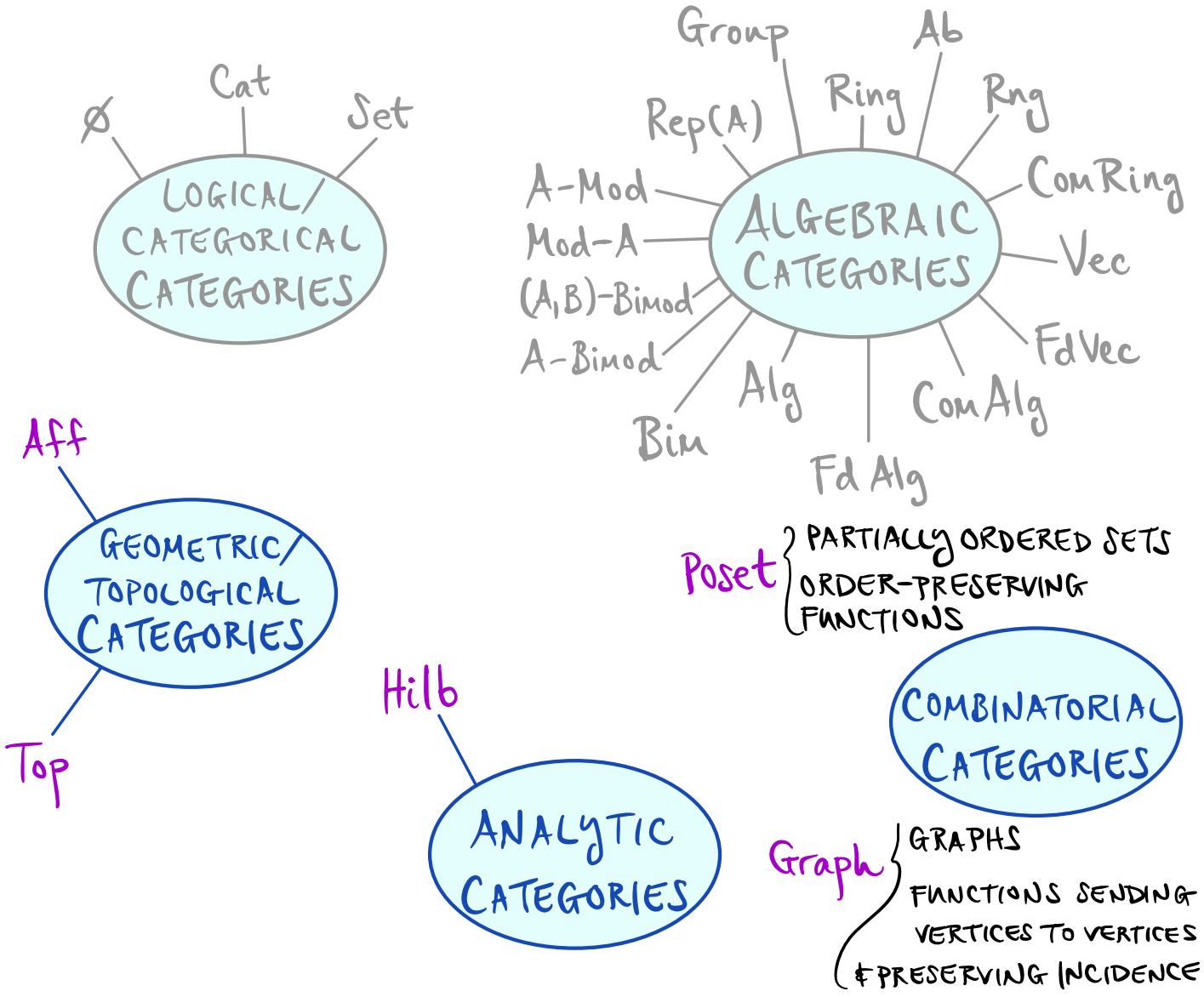

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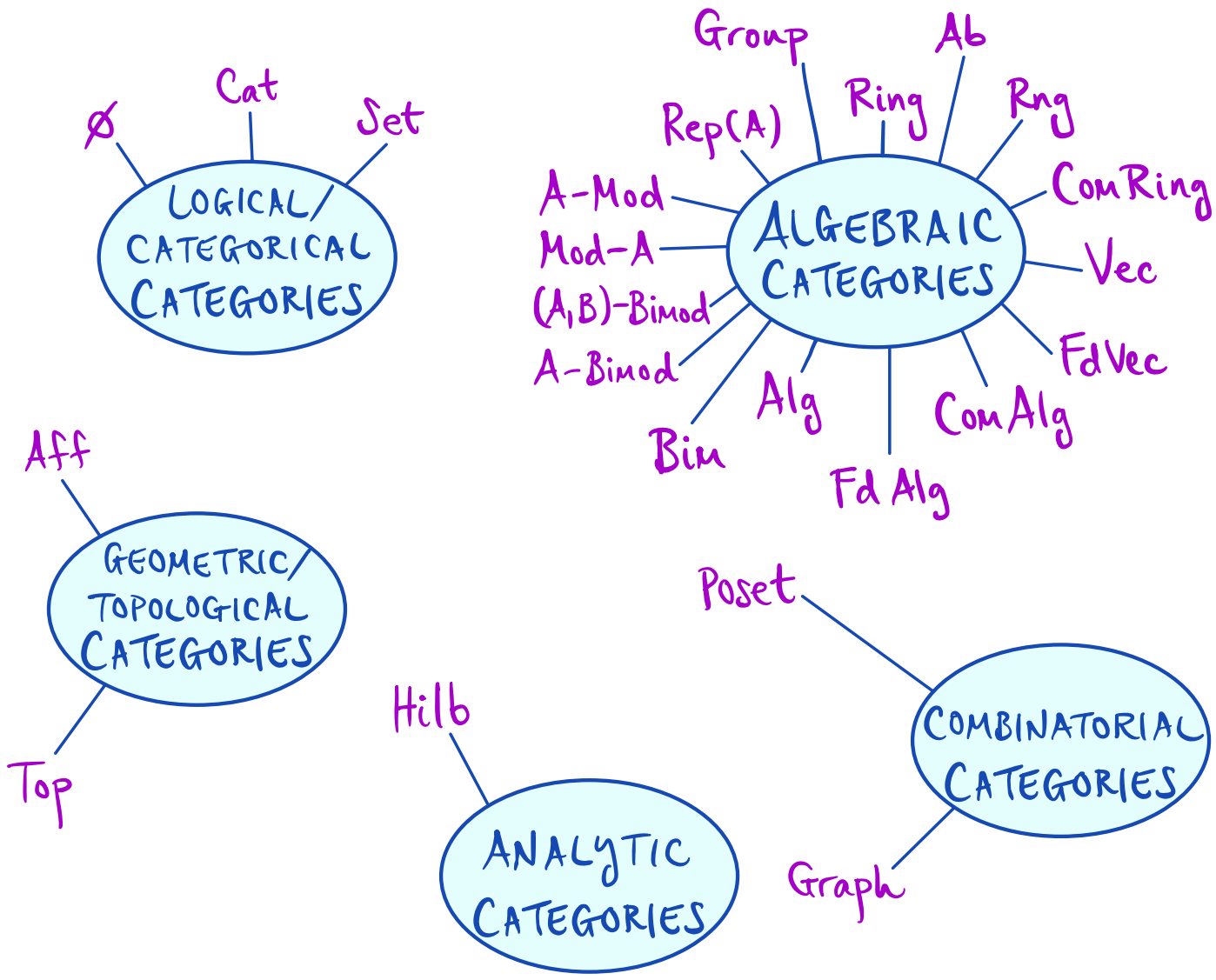

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.....MORE EXAMPLES ??

EXERCISE 2.6 IS THE FOLLOWING A CATEGORY?

80s Music :

- OBJECTS $\equiv$ PERSONS
- $\exists f \in \mathrm{Hom}_{80\text{s Music}}(\text{Person A}, \text{Person B})$

$\iff$ { Person A & Person B BOTH LIKE A CERTAIN TRACK FROM THE 1980s.

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MAKE UP A WEIRD EXAMPLE

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SOME OPERATIONS ON CATEGORIES –

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SOME OPERATIONS ON CATEGORIES –

GIVEN A CATEGORY $\mathcal{C}$,

ITS OPPOSITE CATEGORY $\mathcal{C}^{op}$ IS A CATEGORY DEFINED BY

- $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$
- $\exists f \in \mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) \iff \exists f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$

≡ REVERSE DIRECTION OF MORPHISMS ≡

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- $\mathrm{Ob}(\mathcal{C}^{op}) = \mathrm{Ob}(\mathcal{C})$
- $\exists f \in \mathrm{Hom}_{\mathcal{C}^{op}}(X,Y) \iff \exists f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$

≡ REVERSE DIRECTION OF MORPHISMS ≡

GIVEN CATEGORIES $\mathcal{C}$ AND $\mathcal{C}'$,

ITS PRODUCT CATEGORY $\mathcal{C} \times \mathcal{C}'$ IS A CATEGORY DEFINED BY

- $\mathrm{Ob}(\mathcal{C} \times \mathcal{C}') = \{(X,X') \mid X \in \mathcal{C},\ X' \in \mathcal{C}'\}$
- $\mathrm{Hom}_{\mathcal{C}\times\mathcal{C}'}((X,X'),(Y,Y'))$
  $= \{(g,g') \mid g \in \mathrm{Hom}_{\mathcal{C}}(X,Y),\ g' \in \mathrm{Hom}_{\mathcal{C}'}(X',Y')\}$

≡ THINK ABOUT COMPOSITION OF MORPHISMS ≡

II. UNIVERSAL CONSTRUCTIONS

RECALL UNIVERSAL PROPERTY ...

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X VIA α (OR α')

IS A STRUCTURE $\mathrm{Univ}(X)$

$\ni$. $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X VIA β (OR β')

$\exists!$ STRUCTURE MAP γ (OR γ') MAKING THE DIAGRAM COMMUTE:

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall\beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists!\gamma'"'] \arrow[dr, "\forall\beta'"] \arrow[dr, phantom, shift right=3ex, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'"] & X \end{tikzcd}$$

FORM II

II. UNIVERSAL CONSTRUCTIONS

RECALL UNIVERSAL PROPERTY ...

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X VIA α (OR α')

IS A STRUCTURE $\mathrm{Univ}(X)$

$\ni$. $\forall$ ARBITRARY STRUCTURES $\mathrm{Arb}(X)$ ATTACHED TO X VIA β (OR β')

$\exists!$ STRUCTURE MAP γ (OR γ') MAKING THE DIAGRAM COMMUTE:

$\mathrm{Univ}(X)$ DOESN'T HAVE TO EXIST.

IF EXISTS, THEN IT'S UNIQUE UP TO ISO.

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\forall\beta"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\gamma"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

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FORM II

II. UNIVERSAL CONSTRUCTIONS

WE DON'T HAVE "ELEMENTS" TO WORK WITH IN GENERAL
THIS IS THE MAIN WAY WE'LL DO COMPUTATIONS

RECALL UNIVERSAL PROPERTY ...

GIVEN A GADGET X,

A UNIVERSAL STRUCTURE ATTACHED TO X VIA α_X (OR α'_X)

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FORM II

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

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FORM II

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WE DON'T HAVE "ELEMENTS" TO WORK WITH IN GENERAL THIS IS THE MAIN WAY WE'LL DO COMPUTATIONS

NAMING

Univ(X) IS ATTACHED TO X VIA α_X (OR α'_X)

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

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FORM II

(Univ(X) DOESN'T HAVE TO EXIST.

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SOMETIMES WE ONLY NAME THIS WHEN BUILDING UNIVERSAL CONSTRUCTIONS

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

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SOMETIMES WE NAME BOTH

WHEN BUILDING UNIVERSAL CONSTRUCTIONS

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

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Ex.

WILL BUILD "KERNEL" OF A MORPHISM $f: X \to Y$ $\equiv$ $\begin{cases} \text{OBJECT "}\ker(f)\text{"} \\ \text{EQUIPPED WITH MORPHISM} \\ \ker(f) \xrightarrow{\alpha'_f} X \end{cases}$

II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

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$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

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II. UNIVERSAL CONSTRUCTIONS

UNIVERSAL PROPERTY

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II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{o}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{o}_X"] \\ X \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

≡ THINK ABOUT THE LINK ≡

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

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II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

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FORM II

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

≡ THINK ABOUT THE LINK ≡

GIVEN A CATEGORY $\mathcal{C}$:

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

GIVEN A CATEGORY $\mathcal{C}$:

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

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$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

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FORM II

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A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I				
T				
0				

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

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A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$			
T	$\{\bullet\}$			
0	N/A			

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\, \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

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EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$			
T	$\{\bullet\}$			
0	N/A			

IS A SUBSET OF ANY SET INCLUDING $\emptyset$

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists!\, \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

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A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	← IS A SUBSET OF ANY SET INCLUDING $\emptyset$		
T	$\{\bullet\}$	← ANY FUNCTION MUST SEND ELEMENTS TO ELEMENTS		
0	N/A			

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

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EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$		
T	$\{\bullet\}$	$\{e\}$		
0	N/A	$\{e\}$		

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{o}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{o}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{o} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{o}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\leftsquigarrow \exists! \{e\} \to G$ INCLUSION	
T	$\{\bullet\}$	$\{e\}$	$\leftsquigarrow \exists! G \twoheadrightarrow G/G = \{e\}$	
0	N/A	$\{e\}$		

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{o}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{o}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{o} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{o}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z}$	
T	$\{\bullet\}$	$\{e\}$	0_{RING}	
0	N/A	$\{e\}$	N/A	

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z} \leftsquigarrow \exists! \mathbb{Z} \to R$, $1_{\mathbb{Z}} \mapsto 1_R$	
T	$\{\bullet\}$	$\{e\}$	$0_{\text{RING}} \leftsquigarrow \exists! R \twoheadrightarrow R/R = 0_{\text{RING}}$	
0	N/A	$\{e\}$	N/A	

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] \arrow[ur, phantom, "\circlearrowleft"] & X \end{tikzcd}$$

FORM II

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{o}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{o}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{o} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{o}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z}$	0_{vs}
T	$\{\bullet\}$	$\{e\}$	0_{RING}	0_{vs}
0	N/A	$\{e\}$	N/A	0_{vs}

II. UNIVERSAL CONSTRUCTIONS: INITIAL, TERMINAL, AND ZERO OBJECTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

AN OBJECT $I \in \mathcal{C}$ IS INITIAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM $\vec{0}_X : I \to X$.

$$\begin{tikzcd} I \arrow[d, dashed, "\exists! \vec{0}_X"] \\ X \end{tikzcd}$$

AN OBJECT $T \in \mathcal{C}$ IS TERMINAL IF $\forall X \in \mathcal{C}$ $\exists!$ MORPHISM ${}_X\vec{0} : X \to T$.

$$\begin{tikzcd} X \arrow[d, dashed, "\exists! {}_X\vec{0}"] \\ T \end{tikzcd}$$

A ZERO OBJECT 0 IS AN INITIAL & TERMINAL OBJ.

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z}$	0_{VS}
T	$\{\bullet\}$	$\{e\}$	0_{RING}	0_{VS}
0	N/A	$\{e\}$	N/A	0_{VS}

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd}
X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathrm{Arb}(X)
\end{tikzcd}$$

FORM I

$$\begin{tikzcd}
\mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"] & \\
\mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X
\end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y	PRODUCT OF X AND Y

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y	PRODUCT OF X AND Y
IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING: $\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$	

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y	PRODUCT OF X AND Y
IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING: $\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$	

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] \arrow[dr, phantom, "\circlearrowright"] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$				
$\sqcap$				

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$(+)$ DISJOINT UNION			
$\sqcap$	$\times$ CARTESIAN PRODUCT			

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowright"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowright"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowright"] \arrow[dr, phantom, "\circlearrowright"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION	$\circledast$ FREE PRODUCT		
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT		

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION	$\circledast$ FREE PRODUCT	$\circledast$ FREE PRODUCT	
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT	

II. UNIVERSAL CONSTRUCTIONS: COPRODUCTS AND PRODUCTS

UNIVERSAL PROPERTY

GIVEN A CATEGORY $\mathcal{C}$ & OBJECTS $X, Y \in \mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & \mathrm{Univ}(X) \arrow[d, dashed, "\exists! \gamma_X"] \arrow[dl, phantom, "\circlearrowleft"] \\ & \mathrm{Arb}(X) \end{tikzcd}$$

FORM I

$$\begin{tikzcd} \mathrm{Arb}(X) \arrow[d, dashed, "\exists! \gamma'_X"'] \arrow[dr, "\forall \beta'_X"] \arrow[dr, phantom, "\circlearrowleft"] & \\ \mathrm{Univ}(X) \arrow[r, "\alpha'_X"] & X \end{tikzcd}$$

FORM II

COPRODUCT OF X AND Y IS AN OBJECT $X \sqcup Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha_X : X \to X \sqcup Y$ AND $\alpha_Y : Y \to X \sqcup Y$ SATISFYING:

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists! \gamma"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C & \end{tikzcd}$$

PRODUCT OF X AND Y IS AN OBJECT $X \sqcap Y \in \mathcal{C}$ EQUIPPED WITH MORPHISMS $\alpha'_X : X \sqcap Y \to X$ AND $\alpha'_Y : X \sqcap Y \to Y$ SATISFYING:

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists! \gamma'"] \arrow[dr, "\forall \beta'_Y"] \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowleft"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION	$\circledast$ FREE PRODUCT	$\circledast$ FREE PRODUCT	$\oplus$ DIRECT SUM
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{o}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] \arrow[dl, phantom, "\circlearrowleft"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \arrow[dl, phantom, "\circlearrowleft"] \\ & C & \end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

PRODUCT OF OBJECTS

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{o}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{o}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

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PRODUCT OF OBJECTS

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{o}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

EXAMPLE: $X \sqcup I \cong X$ FOR ANY $X \in \mathcal{C}$.

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

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TERMINAL OBJECT

EXAMPLE: $X \sqcup I \cong X$ FOR ANY $X \in \mathcal{C}$.

PF/ HAVE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\beta_X = \mathrm{id}_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{o}_X = \beta_I"] \\ & X \arrow[ul, phantom, "\circlearrowleft"] \arrow[ur, phantom, "\circlearrowleft"] & \\ & \| & \\ & C & \end{tikzcd}$$

$\therefore\ \gamma\, \alpha_X = \mathrm{id}_X$ STS: $\alpha_X\, \gamma = \mathrm{id}_{X \sqcup I}$

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II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

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$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\mathrm{id}_X"'] \arrow[ddr, bend right, "\alpha_X"'] & X \sqcup I \arrow[d, dashed, "\gamma"] & I \arrow[l, "\alpha_I"'] \arrow[dl, "\vec{o}_X"] \arrow[ddl, bend left, "\alpha_I = \vec{o}_{X \sqcup I}"] \\ & X \arrow[ul, phantom, "\circlearrowleft"] \arrow[ur, phantom, "\circlearrowleft"] \arrow[d, "\alpha_X"] & \\ & X \sqcup I & \end{tikzcd}$$

$\vec{o}_{X \sqcup I} = \alpha_I$ BECAUSE I INITIAL

BECAUSE UNIQUENESS OF $\vec{o}_*$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{o}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C \arrow[ul, phantom, "\circlearrowleft"] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

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PRODUCT OF OBJECTS

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{o}"] \\ T \end{tikzcd}$$

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II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

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INITIAL OBJECT

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COPRODUCT OF OBJECTS

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PRODUCT OF OBJECTS

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\,{}_X\vec{o}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

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ALSO HAVE

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\alpha_X"'] & X \sqcup I \arrow[d, "\mathrm{id}"] & I \arrow[l, "\vec{o}_{X \sqcup I} = \alpha_I"'] \arrow[dl, "\vec{o}_{X \sqcup I}"] \\ & X \sqcup I \arrow[ul, phantom, "\circlearrowleft"] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\,\vec{o}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\,\gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C \arrow[ul, phantom, "\circlearrowleft"] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

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PRODUCT OF OBJECTS

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UNIQUENESS OF γ IN COPRODUCT CONSTRUCTION

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\alpha_X"'] & X \sqcup I \arrow[d, "\mathrm{id}"] & I \arrow[l, "\vec{o}_{X \sqcup I} = \alpha_I"'] \arrow[dl, "\vec{o}_{X \sqcup I}"] \\ & X \sqcup I \arrow[ul, phantom, "\circlearrowleft"] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

$$\begin{tikzcd} X \arrow[r, "\alpha_X"] \arrow[dr, "\forall \beta_X"'] & X \sqcup Y \arrow[d, dashed, "\exists!\, \gamma"] & Y \arrow[l, "\alpha_Y"'] \arrow[dl, "\forall \beta_Y"] \\ & C \arrow[ul, phantom, "\circlearrowleft"] \arrow[ur, phantom, "\circlearrowleft"] & \end{tikzcd}$$

COPRODUCT OF OBJECTS

$$\begin{tikzcd} & P \arrow[dl, "\forall \beta'_X"'] \arrow[d, dashed, "\exists!\, \gamma'"] \arrow[dr, "\forall \beta'_Y"] & \\ X & X \sqcap Y \arrow[l, "\alpha'_X"'] \arrow[r, "\alpha'_Y"] & Y \end{tikzcd}$$

PRODUCT OF OBJECTS

$$\begin{tikzcd} X \arrow[d, dashed, "\exists!\, {}_X\vec{0}"] \\ T \end{tikzcd}$$

TERMINAL OBJECT

<u>LIKEWISE</u> $X \sqcup I \cong X \cong I \sqcup X$ AND $X \sqcap T \cong X \cong T \sqcap X$

FOR ANY $X \in \mathcal{C}$.

II. UNIVERSAL CONSTRUCTIONS: A COMPUTATION

$$\begin{tikzcd} I \arrow[d, dashed, "\exists!\, \vec{0}_X"] \\ X \end{tikzcd}$$

INITIAL OBJECT

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PRODUCT OF OBJECTS

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TERMINAL OBJECT

<u>LIKEWISE</u> $X \sqcup I \cong X \cong I \sqcup X$ AND $X \sqcap T \cong X \cong T \sqcap X$ FOR ANY $X \in \mathcal{C}$.

THINK ABOUT THIS IN THE CONTEXT OF:

EXAMPLES	Set	Group	Ring	Vec
I	$\emptyset$	$\{e\}$	$\mathbb{Z}$	0_{VS}
T	$\{\bullet\}$	$\{e\}$	0_{RING}	0_{VS}
0	N/A	$\{e\}$	N/A	0_{VS}

EXAMPLES	Set	Group	Ring	Vec
$\sqcup$	$\uplus$ DISJOINT UNION	$\circledast$ FREE PRODUCT	$\circledast$ FREE PRODUCT	$\oplus$ DIRECT SUM
$\sqcap$	$\times$ CARTESIAN PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT	$\times$ DIRECT PRODUCT

MATH 466/566
SPRING 2024

CHELSEA WALTON
RICE U.

LECTURE #6

TOPICS:

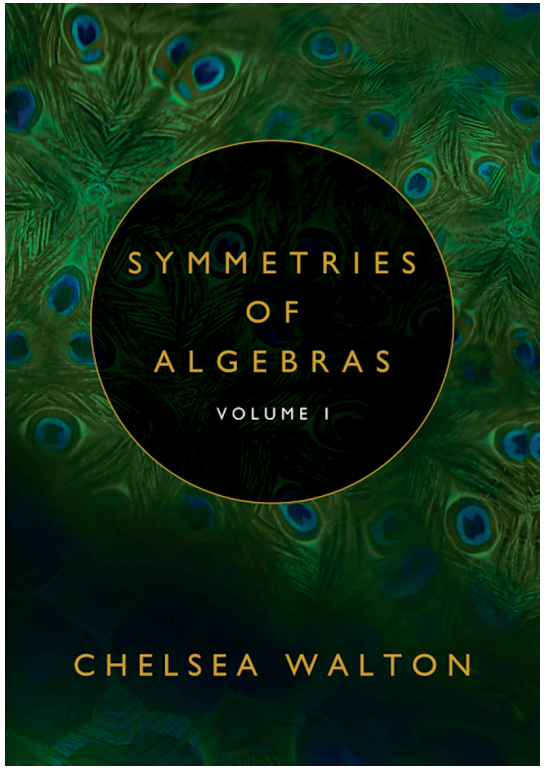
I. CATEGORIES (§2.1)

II. UNIVERSAL CONSTRUCTIONS (§2.2.1) → $I, T, 0$ ✓ ; $\sqcup, \sqcap$ ✓

NEXT TIME: MORE & ABELIAN CATEGORIES

Enjoy this lecture?
You'll enjoy the textbook!

C. Walton's "Symmetries of Algebras, Volume 1" (2024)



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&
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Lecture #6 keywords: category, coproduct of objects, initial object, morphism, product of objects, object, terminal object, universal construction, zero object